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UNRAMIFIEDNESS OF GALOIS REPRESENTATIONS ATTACHED TO WEIGHT 1 HILBERT MODULAR EIGENFORMS MOD p

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Abstract. The main result of this article states that the Galois representation attached to a Hilbert modular eigenform over $\overline{\mathbb{F}}_p$ of parallel weight 1 and level prime to p is unramified above p. This includes the important case of eigenforms that do not lift to Hilbert modular forms in characteristic 0 of parallel weight 1. The proof is based on the observation that parallel weight 1 forms in characteristic p embed into the ordinary part of parallel weight p in two different ways per place above p, namely via 'partial' Frobenius operators.

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1. Introduction

Let p be any prime number. Under Serre's Modularity Conjecture [33] (a theorem of Khare and Wintenberger [25]) Hecke eigenforms for $\mathrm{GL}_2(\mathbb{Q})$ with coefficients in $\overline{\mathbb{F}}_p$ correspond to odd semisimple representations $G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Using 'minimal weights', as suggested by Serre and taken up by Edixhoven [15], Katz modular forms of weight 1 and level prime to p give rise to Galois representations that have the special property of being unramified at p, and the converse is also true (at least when $p > 2$), see [7], [22], [38].

In this article we generalise this result to Hilbert modular forms over any totally real number field F. Let G_F denote the absolute Galois group of F and for each prime $\mathfrak{q}$ of F we fix a decomposition group $D_{\mathfrak{q}}$ of G_F at $\mathfrak{q}$ and denote by $\mathrm{Frob}_{\mathfrak{q}}$ an arithmetic Frobenius.

Theorem 1.1. *Let f be a Hilbert modular form over $\overline{\mathbb{F}}_p$ of parallel weight 1 and level $\mathfrak{n}$ which is relatively prime to p. Assume that f is a common eigenvector for the Hecke operators $T_{\mathfrak{q}}$ for all $\mathfrak{q}$ outside a finite set Σ of primes of F containing those dividing $\mathfrak{n}$. Then there exists a continuous semi-simple representation*

$$\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is unramified outside $\mathfrak{n}$, and satisfies that for all primes $\mathfrak{q} \notin \Sigma$ the trace of $\rho_f(\mathrm{Frob}_{\mathfrak{q}})$ equals the eigenvalue of $T_{\mathfrak{q}}$ on f.

The main content in the above theorem is the unramifiedness of ρ_f at all primes $\mathfrak{p}$ dividing p. This theorem was proved independently by Emerton, Reduzzi and Xiao [17] under the additional assumptions that p is inert in F and the Hecke

polynomial at p has two distinct roots: the p-distinguished case. Note that if Σ does not contain a prime $\mathfrak{p}$ dividing p, then the theorem also states that the trace of $\mathrm{Frob}_{\mathfrak{p}}$ under the representation is the eigenvalue of $T_{\mathfrak{p}}$ on f.

Hecke operators. A substantial part of the paper is devoted to constructing for $\mathfrak{p}$ a prime dividing p a Hecke operator $T_{\mathfrak{p}}^{(k)}$ acting of the space of Hilbert modular forms over $\mathbb{F}_p$ of parallel weight $k \geq 1$ and level which is relatively prime to p. Although $T_{\mathfrak{p}}^{(k)}$ is uniquely determined by its action on q-expansions, hence *a fortiori* by its action on p-adic modular forms (over the ordinary locus), its definition in the current state of the knowledge seems to require some geometric input from the non-ordinary locus of the Hilbert modular variety. Our construction relies on a fine geometric result from [17] stating roughly that for our purposes the non-ordinary locus can be ignored (see §3.3.4). A different construction of $T_{\mathfrak{p}}^{(k)}$ was recently announced by V. Pilloni.

A specific feature of our treatment is that we compute the action of $T_{\mathfrak{p}}^{(k)}$ on adelic q-expansions, which we expect to be also useful for questions on Hilbert modular forms over $\overline{\mathbb{F}}_p$ beyond the ones treated in this article.

Theorem 1.2 (Effect of $T_{\mathfrak{p}}$ on q-expansions)**.** *For any Hilbert modular form f over F of parallel weight $k \geq 1$ and prime to p level, and for any prime $\mathfrak{p}$ of F dividing p one has:*

$$\begin{aligned} a((0), T_{\mathfrak{p}}^{(k)} f) &= a((0), f)[\mathfrak{p}] + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{k-1} a((0), \langle \mathfrak{p} \rangle f)[\mathfrak{p}^{-1}], \text{ and} \\ a(\mathfrak{r}, T_{\mathfrak{p}}^{(k)} f) &= a(\mathfrak{p}\mathfrak{r}, f) + \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{k-1} a(\mathfrak{r}/\mathfrak{p}, \langle \mathfrak{p} \rangle f) \text{ for all } (0) \neq \mathfrak{r} \subset \mathfrak{o}. \end{aligned} \tag{1}$$

Here $a(\mathfrak{r}, f)$ denote the coefficients in the adelic q-expansion of f (see §2.6) and $\langle \mathfrak{p} \rangle$ denotes the diamond operator (see §3.1).

Liftable vs. non-liftable forms of weight 1. Extending earlier work of Deligne and Serre [10], Ohta [28], Rogawski and Tunnell [30] attach to any complex holomorphic parallel weight 1 Hilbert modular newform a 2-dimensional irreducible totally odd Artin representation of G_F which is unramified outside the level of the form. In a series of works by Kassaei, Pilloni, Sasaki, Stroh and Tian, building on a strategy of Buzzard and Taylor [3], it has been shown that the converse also holds. However, since Hilbert modular forms over $\overline{\mathbb{F}}_p$ do not necessarily lift to characteristic 0 in the same weight, the main result of the present article cannot be deduced form those works.

In his forthcoming PhD thesis [37], Jasper Van Hirtum implements an algorithm for computing spaces of Hilbert modular forms of parallel weight 1, which we expect to provide explicit examples of non-liftable parallel weight 1 Hilbert eigenforms. In the perspective of the inverse Galois problem, this will provide examples of Galois extensions of F with group $\mathrm{PGL}_2(\mathbb{F}_p)$ the unramifiedness above p of which would follow from our main theorem, but would be very hard to check without it since

one would in general not be able to write down an explicit polynomial giving the extension. Such fields are also expected to have a small root discriminant.

Relation to conjectures of Serre and Buzzard-Diamond-Jarvis. We see our main result as an important step towards a Serre type modularity conjecture over totally real number fields F which includes as fine arithmetic information as possible (in particular, on ramification). Serre's Modularity Conjecture has been generalised to general F by Buzzard, Diamond and Jarvis [2] *et al.*, and the weight part of this conjecture has already been established (see [20]). However, information on the unramifiedness above p on the one hand, and on parallel weight 1 forms on the other hand is not tangible in the Buzzard-Diamond-Jarvis conjecture due to its formulation in terms of Betti cohomology.

In analogy with the situation over $\mathbb{Q}$ it is conjectured that totally odd, irreducible representations $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ which are unramified above p correspond exactly to parallel weight 1 eigenforms over $\overline{\mathbb{F}}_p$ of level prime to p. Our main theorem establishes one direction of this correspondence in full generality. The converse has been proved in the work on companion forms [19] of Gee and Kassaei under the assumptions that p is odd, ρ is modular and p-distinguished, and some other milder hypotheses.

$R = \mathbb{T}$-theorems. We expect that for any f as in Theorem 1.1 for which ρ_f is irreducible our result can be extended into proving the unramifiedness above p of the Galois representation with coefficients in the parallel weight 1 Hecke algebra $\mathbb{T}$ over $\overline{\mathbb{F}}_p$ of level $\mathfrak{n}$ localised at ρ_f. It is further expected that $\mathbb{T}$ is in fact the universal deformation Artinian $\overline{\mathbb{F}}_p$-algebra of ρ_f unramified outside $\mathfrak{n}$ with prescribed local properties at primes dividing $\mathfrak{n}$. This has been proved for $F = \mathbb{Q}$ and $p > 2$ by Calegari and Geraghty [4], who even establish more, namely that all unramified at p minimal deformations of ρ_f are modular of weight 1.

Generalising the approach of Calegari and Geraghty to an arbitrary totally real F needs as a crucial ingredient the unramifiedness above p of the Galois representations contributing to certain higher degree coherent cohomology groups (not only to those of degree zero corresponding to modular forms). This seems to require new ideas (see [17] for a partial result in the real quadratic case).

Sketch of the proof. Let us now give a short sketch of the proof of the main theorem and highlight some of its main features. In this article we will exclusively work with parallel weight and so the word 'parallel' will henceforth be dropped.

The definition of Hilbert modular forms used is in the spirit of Wiles, as compared to the theory of Katz. By this we mean, as explained in Section 2, that we use the Hilbert modular forms living on the Hilbert modular variety classifying Hilbert-Blumenthal abelian varieties modulo the action of the totally positive units on polarisations. In fact, working with modular forms à la Wiles is necessary in order to have a good Hecke theory, *i.e.*, seems to be the right setting for working with Galois representations. Moreover, over $\mathbb{C}$ those are precisely the automorphic forms

on GL_2 studied by Shimura in [34]. The technical complications arising from the fact that the relevant Hilbert modular variety is only a finite quotient of the fine moduli scheme are properly addressed. The literature is not as complete and as uniform in terms of definitions as one would wish; we have nonetheless tried to include as many and as precise references as possible.

Our general strategy is to first go to weight p where we exploit the subtle phenomenon of 'doubling' for Hecke algebras (failure to strong multiplicity one in characteristic p; see [38]). Whereas the overall strategy of proof is very similar to the one in [38], the actual techniques used here are quite different since many of the arguments in *loc. cit.* seem to be specific to $F = \mathbb{Q}$.

Let us start with a weight 1 Hilbert modular form f over $\overline{\mathbb{F}}_p$ of level prime to p which is an eigenform for all Hecke operators $T_{\mathfrak{q}}$ with eigenvalues $\lambda_{\mathfrak{q}}$. Let ρ_f be the attached Galois representation. Via multiplication by the Hasse invariant h, which has weight $p-1$ and q-expansions equal to 1, f can be mapped into weight p. Let W be the subspace spanned by the images of hf under all Hecke operators above p; this is the same as the space spanned by the images of f under all partial Frobenius operators, which we define in this article based on the Hecke operators $T_{\mathfrak{p}}$ for $\mathfrak{p}$ dividing p. The properties of W which we need are deduced from the interplay between the partial Frobenius operators and the Hecke operators above p.

Let us fix a prime ideal $\mathfrak{p}$ of F above p and sketch why ρ_f is unramified at $\mathfrak{p}$. For all other primes $\mathfrak{p}'$ dividing p, let $\alpha_{\mathfrak{p}'}$ be one of the (possibly two) eigenvalues of $T_{\mathfrak{p}'}$ acting on W. Consider the subspace $W_{\mathfrak{p}}$ of W on which $T_{\mathfrak{p}'}$ acts as multiplication by $\alpha_{\mathfrak{p}'}$ for all $\mathfrak{p}'$ above p different from $\mathfrak{p}$. It turns out that $W_{\mathfrak{p}}$ is 2-dimensional and $T_{\mathfrak{p}}$ acts on it with minimal polynomial $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p}) = (X - \alpha_{\mathfrak{p}})(X - \beta_{\mathfrak{p}})$, where ε is the central character of f. Multiplying further by high enough powers of the Hasse invariant, this 2-dimensional space can be lifted into the ordinary part in characteristic 0.

If $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ are distinct (that is the *p-distinguished case*), then Wiles' local description of the Galois representation attached to ordinary modular forms shows that ρ_f has two quotients that are unramified at $\mathfrak{p}$, on which $\mathrm{Frob}_{\mathfrak{p}}$ acts through $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$, respectively, implying that ρ_f is unramified at $\mathfrak{p}$.

If $\alpha_{\mathfrak{p}} = \beta_{\mathfrak{p}}$, then a more involved argument, based on the fact that the action of $T_{\mathfrak{p}}$ on $W_{\mathfrak{p}}$ is not semi-simple, yields the desired result (see §4.4).

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2. Geometric Hilbert modular forms

In this section we recall and adapt some results from Andreatta-Goren [1], Chai [6], Deligne-Pappas [9], Dimitrov [11, 13], Dimitrov-Tilouine [14], Hida [23], Kisin-Lai [26] and Rapoport [31].

Let F be a totally real field of degree $d \geq 2$, ring of integers $\mathfrak{o}$ and different $\mathfrak{d}$. Fix an ideal $\mathfrak{n} \subset \mathfrak{o}$ (the level). For any fractional ideal $\mathfrak{c}$ of F we denote by $\mathfrak{c}_+ = \mathfrak{c} \cap F_+^\times$ the cone of its totally positive elements. The set of equivalence classes of $\mathfrak{c}$ modulo the action of $F_+^\times$ can be naturally identified with the narrow class group $\mathcal{C}\ell_F^+$ of F. By abuse of notation, we will also denote by $\mathcal{C}\ell_F^+$ a fixed set of representatives.

2.1. Hilbert modular varieties.

Recall that a Hilbert-Blumenthal abelian variety over a scheme S is an abelian scheme $\pi : A \to S$ of relative dimension d together with an injection $\mathfrak{o} \hookrightarrow \mathrm{End}(A/S)$. The dual abelian variety of A is denoted by $A^\vee$.

A $\mathfrak{c}$-polarisation on a Hilbert-Blumenthal abelian variety A/S is an $\mathfrak{o}$-linear isomorphism

$$\lambda : A \otimes_{\mathfrak{o}} \mathfrak{c} \xrightarrow{\sim} A^\vee,$$

such that the induced isomorphism $\mathrm{Hom}_{\mathfrak{o}}(A, A \otimes_{\mathfrak{o}} \mathfrak{c}) \simeq \mathrm{Hom}_{\mathfrak{o}}(A, A^\vee)$ sends $\mathfrak{c}$ (resp. $\mathfrak{c}_+$) onto the $\mathfrak{o}$-module of symmetric elements $\mathcal{P}(A)$ (resp. onto the cone of polarisations $\mathcal{P}(A)_+$).

A $\mu_{\mathfrak{n}}$-level structure on a Hilbert-Blumenthal abelian variety A/S is an $\mathfrak{o}$-linear closed immersion $\mu : \mu_{\mathfrak{n}} \otimes \mathfrak{d}^{-1} \hookrightarrow A$ of group schemes over S, where $\mu_{\mathfrak{n}}$ denotes the reduced subscheme of $\mathbb{G}_m \otimes \mathfrak{d}^{-1}$ defined as the intersection of the kernels of multiplication by elements of $\mathfrak{n}$. Assume that

(2) $$\mathfrak{n} \text{ divides neither } 2, \text{ nor } 3, \text{ nor the discriminant } \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{d}).$$

Then the functor assigning to a $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$-scheme S the set of isomorphism classes of tuples (A, λ, μ) as above is representable by a geometrically connected quasi-projective scheme $X_1^1(\mathfrak{c}, \mathfrak{n}) \to \mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$, which is a relative complete intersection, flat and has normal fibres (see [9, Theorem 2.2 and Corollary 2.3]). We denote by $\pi : \mathcal{A}(\mathfrak{c}) \to X_1^1(\mathfrak{c}, \mathfrak{n})$ the universal abelian variety.

Recall that the Rapoport locus of $X_1^1(\mathfrak{c}, \mathfrak{n})$, defined as the locus where the sheaf $\pi_* \Omega^1_{\mathcal{A}(\mathfrak{c})/X_1^1(\mathfrak{c},\mathfrak{n})}$ is locally free of rank one over $\mathfrak{o} \otimes \mathcal{O}_{X_1^1(\mathfrak{c},\mathfrak{n})}$, is an open sub-scheme,

which is smooth over $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d})}]$, and its complement is contained in fibres in characteristics dividing $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d})$ and has codimension at least 2 in those fibres.

Consider the following invertible sheaf on $X_1^1(\mathfrak{c},\mathfrak{n})$:

$$\underline{\omega}^1 := \det(\pi_*\Omega^1_{\mathcal{A}(\mathfrak{c})/X_1^1(\mathfrak{c},\mathfrak{n})}) = \wedge^d_{\mathcal{O}_{X_1^1(\mathfrak{c},\mathfrak{n})}} \pi_*\Omega^1_{\mathcal{A}(\mathfrak{c})/X_1^1(\mathfrak{c},\mathfrak{n})}.$$

Make the following, stronger than (2), assumption:

(3) $\mathfrak{n}$ is divisible by a prime number q which splits completely in $F(\sqrt{\epsilon}\,|\,\epsilon \in \mathfrak{o}_+^\times)$ and $q \equiv -1 \pmod{4\ell}$ for all prime numbers ℓ such that $[F(\mu_\ell):F] = 2$.

Under the assumption (3) the finite group

$$E := \mathfrak{o}_+^\times/\{\epsilon \in \mathfrak{o}^\times | \epsilon - 1 \in \mathfrak{n}\}^2 \tag{4}$$

acts properly and discontinuously on $X_1^1(\mathfrak{c},\mathfrak{n})$ by

$$[\epsilon] : (A,\lambda,\mu)/S \mapsto (A,\epsilon\lambda,\mu)/S,$$

yielding an étale morphism $\phi : X_1^1(\mathfrak{c},\mathfrak{n}) \to X_1(\mathfrak{c},\mathfrak{n})$ with group E ([12, Lemma 2.1]). It follows that $X_1(\mathfrak{c},\mathfrak{n}) \to \mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$ is geometrically connected, flat with normal fibres, and smooth over $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d})}]$. Moreover $\underline{\omega}^1$ descends to an invertible sheaf on $X_1(\mathfrak{c},\mathfrak{n})$, denoted $\underline{\omega}$ (see [14, §4]), such that its pull-back is $\underline{\omega}^1$.

Let $X_1^1(\mathfrak{n}) = \coprod_{\mathfrak{c}} X_1^1(\mathfrak{c},\mathfrak{n})$ and $X_1(\mathfrak{n}) = \coprod_{\mathfrak{c}} X_1(\mathfrak{c},\mathfrak{n})$, where $\mathfrak{c}$ runs over the fixed set of representatives of $\mathcal{C}\ell_F^+$. The latter is called the Hilbert modular variety of level $\Gamma_1(\mathfrak{n})$, since its complex points can be identified with those of a Shimura variety for $\mathrm{GL}_2(F)$. Note that the points of $X_1(\mathfrak{n})$ over any $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$-scheme S are naturally given by isomorphism classes of Hilbert-Blumenthal abelian varieties A/S endowed with a $\mu_{\mathfrak{n}}$-level structure and satisfying the Deligne-Pappas condition that the induced morphism $A \otimes_{\mathfrak{o}} \mathcal{P}(A) \to A^\vee$ is an isomorphism. Thus $X_1(\mathfrak{n})$ is independent of the choice of representatives in $\mathcal{C}\ell_F^+$.

2.2. **Hilbert modular forms of parallel weight.** Fix an integer $k \geq 1$. For any ideal $\mathfrak{n} \subset \mathfrak{o}$ one can define as in Katz [24] the space of $\mathfrak{c}$-polarised Hilbert modular forms of weight k, level $\mathfrak{n}$ and coefficients in a $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$-algebra R. As observed in [26, §1.4], under the assumption (2) this space is given by:

$$\mathrm{H}^0(X_1^1(\mathfrak{c},\mathfrak{n}) \times \mathrm{Spec}(R), (\underline{\omega}^1)^{\otimes k}).$$

In this paper, we will however reserve the name of Hilbert modular forms to those forms which are invariant under the action of the finite group E defined in (4). In fact, in characteristic 0, those are the only forms for which there is a satisfactory Hecke theory allowing to attach Galois representations to eigenforms.

Note that it suffices to prove Theorem 1.1 under assumption (3), which we can and will always assume in the sequel. The space of Hilbert modular forms of weight

k, level $\mathfrak{n}$ and coefficients in a $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$-algebra R, is then defined as the R-module:

$$M_k(\mathfrak{n};R) := \mathrm{H}^0(X_1(\mathfrak{n}) \times \mathrm{Spec}(R), \underline{\omega}^{\otimes k}). \tag{5}$$

We have the following decomposition $M_k(\mathfrak{n};R) = \bigoplus_{\mathfrak{c}\in\mathcal{C}\ell_F^+} M_k(\mathfrak{c},\mathfrak{n};R)$ with $\mathfrak{c}$ running over the fixed set of representatives of $\mathcal{C}\ell_F^+$, where

$$M_k(\mathfrak{c},\mathfrak{n};R) = \mathrm{H}^0(X_1(\mathfrak{c},\mathfrak{n}) \times \mathrm{Spec}(R), \underline{\omega}^{\otimes k}).$$

2.3. **Minimal compactification.** Following [6, §4] (see also [13]) one defines the minimal compactification of $X_1^1(\mathfrak{c},\mathfrak{n})$ as the $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$-scheme given by

$$\overline{X_1^1(\mathfrak{c},\mathfrak{n})} = \mathrm{Proj}\left(\bigoplus_{k\geq 0} \mathrm{H}^0(X_1^1(\mathfrak{c},\mathfrak{n}), (\underline{\omega^1})^{\otimes k})\right).$$

The scheme $\overline{X_1^1(\mathfrak{c},\mathfrak{n})}$ is projective, normal, hence flat (see [16, IV.6.8.1]) over $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$ and contains $X_1^1(\mathfrak{c},\mathfrak{n})$ as a dense open sub-scheme whose complement is finite over $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$.

The action of the finite group E on $X_1^1(\mathfrak{c},\mathfrak{n})$ extends to an action on $\overline{X_1^1(\mathfrak{c},\mathfrak{n})}$ and the minimal compactification $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ of $X_1(\mathfrak{c},\mathfrak{n})$ is defined as the quotient. The $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$-scheme $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ is projective, normal, flat over $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n})}]$ and contains $X_1(\mathfrak{c},\mathfrak{n})$ as a dense open sub-scheme. Alternatively, one can also define $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ as $\mathrm{Proj}\left(\bigoplus_{k\geq 0} \mathrm{H}^0(X_1(\mathfrak{c},\mathfrak{n}), \underline{\omega}^{\otimes k})\right)$.

Lemma 2.1. *The invertible sheaf $\underline{\omega}$ on $X_1(\mathfrak{c},\mathfrak{n})$ extends to an ample invertible sheaf $\overline{\underline{\omega}}$ on $\overline{X_1(\mathfrak{c},\mathfrak{n})}$.*

Proof. As explained in [26, §1.8], the arguments of [6, §4.4] go over unchanged to our setting and prove that the invertible sheaf $\underline{\omega^1}$ on $X_1^1(\mathfrak{c},\mathfrak{n})$ extends to an ample invertible sheaf $\overline{\underline{\omega^1}}$ on $\overline{X_1^1(\mathfrak{c},\mathfrak{n})}$. By [14, Theorem 8.6(vi)], $\underline{\omega}$ extends to an invertible sheaf $\overline{\underline{\omega}}$ on $\overline{X_1(\mathfrak{c},\mathfrak{n})}$. To show that $\overline{\underline{\omega}}$ is ample, consider the commutative diagram:

$$\begin{tikzcd} X_1^1(\mathfrak{c},\mathfrak{n}) \arrow[r] \arrow[d, "\phi"] & \overline{X_1^1(\mathfrak{c},\mathfrak{n})} \arrow[d, "\bar\phi"] \\ X_1(\mathfrak{c},\mathfrak{n}) \arrow[r] & \overline{X_1(\mathfrak{c},\mathfrak{n})} \end{tikzcd},$$

where $\phi : X_1^1(\mathfrak{c},\mathfrak{n}) \to X_1(\mathfrak{c},\mathfrak{n})$ is finite étale with group E, whereas $\bar\phi : \overline{X_1^1(\mathfrak{c},\mathfrak{n})} \to \overline{X_1(\mathfrak{c},\mathfrak{n})}$ is a finite surjective morphism of normal schemes. By [16, II.6.6.2] the norm $\mathcal{N}_{\bar\phi}(\overline{\underline{\omega^1}})$ is an ample invertible sheaf on $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ (note that whereas condition

(I) of *loc. cit.* may not hold since $\bar{\phi}$ is not necessarily flat, condition (IIbis) is always fulfilled since $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ is normal). Moreover one has

$$\mathcal{N}_{\bar{\phi}}(\underline{\overline{\omega}^1})_{|X_1(\mathfrak{c},\mathfrak{n})} = \mathcal{N}_{\phi}(\underline{\omega}^1) = \mathcal{N}_{\phi}(\phi^*\underline{\omega}) = \underline{\omega}^{\otimes|E|},$$

where the final equality follows from [16, II.6.5.2.4]. Since $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ is normal and the complement of $X_1(\mathfrak{c},\mathfrak{n})$ has codimension $d \geq 2$, one deduces that $\mathcal{N}_{\bar{\phi}}(\underline{\overline{\omega}^1}) = \underline{\overline{\omega}}^{\otimes|E|}$. Therefore $\underline{\overline{\omega}}^{\otimes|E|}$ is ample, which implies that $\underline{\overline{\omega}}$ is ample too. $\square$

The complement of $X_1(\mathfrak{c},\mathfrak{n})$ in $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ consists of closed points, called cusps.

Amongst these one has the 'infinity' cusps $\infty(\mathfrak{c};\mathfrak{b}) : \mathrm{Spec}(\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]) \to \overline{X_1(\mathfrak{c},\mathfrak{n})}$, with $\mathfrak{b}$ running over a set of representatives of $\mathcal{C}\ell_F$, the standard infinity cusp being $\infty(\mathfrak{c}) = \infty(\mathfrak{c};\mathfrak{o})$. By [13, §4] the inverse image of $\infty(\mathfrak{c},\mathfrak{b})$ under $\bar{\phi}$ in $\overline{X_1^1(\mathfrak{c},\mathfrak{n})}$ consists of a unique point and the Tate object at that cusp is given by $(\mathbb{G}_m \otimes_{\mathfrak{o}} \mathfrak{b}^{-1}\mathfrak{c}^{-1}\mathfrak{d}^{-1})/q^{\mathfrak{b}}$.

Let R be a $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$-algebra. The Koecher Principle (see [31, 4.9] and [13, Theorem 8.3]) implies that the natural restriction map

$$\text{(6)} \quad \mathrm{H}^0(\overline{X_1(\mathfrak{c},\mathfrak{n})} \times \mathrm{Spec}(R), \underline{\overline{\omega}}^{\otimes k}) \xrightarrow{\sim} \mathrm{H}^0(X_1(\mathfrak{c},\mathfrak{n}) \times \mathrm{Spec}(R), \underline{\omega}^{\otimes k}) = M_k(\mathfrak{c};\mathfrak{n};R).$$

is an isomorphism. Note that the arguments from *loc. cit.* are purely at ∞, hence also apply to R in which the discriminant is not necessarily invertible.

2.4. **Ampleness and lifting.** Fix a prime number p, relatively prime to $\mathfrak{n}$.

The construction of a Galois representation for a normalised Hilbert modular eigenform of weight 1 over $\overline{\mathbb{F}}_p$ is achieved by constructing a lifting in characteristic 0 in some, possibly higher, weight. The existence of such lifts with special properties relative to the Hecke operators $T_{\mathfrak{p}}$ at primes $\mathfrak{p}$ dividing p, namely being ordinary, is an important ingredient in the proof of our main theorem, allowing us to apply theorems of Hida and Wiles describing the local behaviour at $\mathfrak{p}$ of the p-adic Galois representations attached to those lifts. Contrary to the situation for classical modular forms over $\mathbb{Q}$, we are not aware of a precise way to control the weight of the lift. If the weight is at least 3, Stroh and Lan-Suh have proved independently that one can lift cusp forms under the condition that p is at least d and does not divide $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d})$ (see [27]).

We now prove the lifting result we need with a method going back to Katz based on Serre's vanishing theorem in coherent cohomology (compare with [1, 11.9]).

Lemma 2.2. *There exists an integer k_0 depending on $\mathfrak{n}$, such that for all $k > k_0$ there is a natural Hecke equivariant isomorphism:*

$$M_k(\mathfrak{n};\mathbb{Z}_p) \otimes \mathbb{F}_p \simeq M_k(\mathfrak{n};\mathbb{F}_p).$$

Proof. Fix a fractional ideal $\mathfrak{c}$ and let $X = X_1(\mathfrak{c},\mathfrak{n}) \times \mathrm{Spec}(\mathbb{Z}_p)$. The projective variety $\overline{X} = \overline{X_1(\mathfrak{c},\mathfrak{n})} \times \mathrm{Spec}(\mathbb{Z}_p)$ is normal, hence flat over $\mathbb{Z}_p$. Since $\underline{\overline{\omega}}^{\otimes k}$ is a locally free $\mathcal{O}_{\overline{X}}$-module, one gets a short exact sequence of sheaves on $\overline{X}$:

$$0 \to \underline{\overline{\omega}}^{\otimes k} \xrightarrow{p\cdot} \underline{\overline{\omega}}^{\otimes k} \to \underline{\overline{\omega}}^{\otimes k}_{\mathbb{F}_p} \to 0,$$

yielding, using Koecher's principle (6), a long exact sequence in cohomology:

$$0 \to M_k(\mathfrak{c},\mathfrak{n};\mathbb{Z}_p) \xrightarrow{p\cdot} M_k(\mathfrak{c},\mathfrak{n};\mathbb{Z}_p) \to \mathrm{H}^0(\overline{X},\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) \to \mathrm{H}^1(\overline{X},\underline{\omega}^{\otimes k}).$$

Since $\underline{\omega}_{\mathbb{F}_p}^{\otimes k}$ has its support in $\overline{X}\times\mathrm{Spec}(\mathbb{F}_p)$, and using again (6) one gets

$$\mathrm{H}^0(\overline{X},\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) = \mathrm{H}^0(\overline{X}\times\mathrm{Spec}(\mathbb{F}_p),\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) = M_k(\mathfrak{c},\mathfrak{n};\mathbb{F}_p).$$

Finally by a theorem due to Serre [16, III.2.2.1], since $\overline{X}$ is projective over $\mathbb{Z}_p$ and $\underline{\omega}$ is an ample invertible sheaf, there exists an integer k_0, such that $\mathrm{H}^1(\overline{X},\underline{\omega}^{\otimes k})$ vanishes for all $k > k_0$. Letting $\mathfrak{c}$ run over the fixed set of representatives of $\mathcal{C}\ell_F^+$ yields the desired result. □

2.5. **Geometric q-expansion.** Let R be a $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$-algebra. By [13, Theorem 8.6] the local completed ring of $\overline{X_1(\mathfrak{c},\mathfrak{n})}\times\mathrm{Spec}(R)$ along $\infty(\mathfrak{c})$ is given by

$$M_\infty(\mathfrak{c};R) := \left\{ \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi q^\xi \mid a_\xi\in R \text{ and } \ a_{\epsilon\xi} = a_\xi, \forall\epsilon\in\mathfrak{o}_+^\times \right\}. \tag{7}$$

Furthermore by *loc. cit.* the local completed ring of $\overline{X_1(\mathfrak{c},\mathfrak{n})}\times\mathrm{Spec}(R)$ along $\infty(\mathfrak{c};\mathfrak{b})$ is given by $M_\infty(\mathfrak{c}\mathfrak{b}^2;R)$.

Since $\underline{\omega}$ can be trivialised over the formal neighbourhood $\mathcal{X}(\mathfrak{c}) := \mathrm{Spf}(M_\infty(\mathfrak{c};R))$ of $\infty(\mathfrak{c})$, (6) yields the *geometric q-expansion* map

$$M_k(\mathfrak{c},\mathfrak{n};R) \to M_\infty(\mathfrak{c};R),\ \ f_\mathfrak{c} \mapsto \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi(f)q^\xi, \tag{8}$$

which due to its local definition factors through the map

$$\mathrm{H}^0(U,\underline{\omega}^{\otimes k}) \to M_\infty(\mathfrak{c};R),\ \ f\mapsto f_\mathfrak{c} = \sum_{\xi\in\mathfrak{c}_+\cup\{0\}} a_\xi(f_\mathfrak{c})q^\xi, \tag{9}$$

for any open subscheme $U\to\mathrm{Spec}(R)$ of $\overline{X_1(\mathfrak{c},\mathfrak{n})}\times\mathrm{Spec}(R)$ containing $\infty(\mathfrak{c})$.

Proposition 2.3 (q-expansion principle)**.** *The map* (9) *is injective. Moreover, for any* $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$*-subalgebra* $R'\subset R$ *and any open subscheme* $U\to\mathrm{Spec}(R')$ *of* $\overline{X_1(\mathfrak{c},\mathfrak{n})}$ *containing* $\infty(\mathfrak{c})$*, one has*

$$\mathrm{H}^0(U\times_{\mathrm{Spec}(R')}\mathrm{Spec}(R),\underline{\omega}^{\otimes k})\cap M_\infty(\mathfrak{c};R') = \mathrm{H}^0(U,\underline{\omega}^{\otimes k}).$$

In particular $M_k(\mathfrak{c},\mathfrak{n};R)\cap M_\infty(\mathfrak{c};R') = M_k(\mathfrak{c},\mathfrak{n};R')$.

Proof. Since the geometric fibres of $X_1(\mathfrak{c},\mathfrak{n})\to\mathrm{Spec}(\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}])$ are normal, the argument of [31, Theorem 6.1] implies that they are irreducible. Hence the geometric fibres of $U\to\mathrm{Spec}(\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}])$ are irreducible too and the above claims follow by exactly the same arguments as in [31, Theorem 6.7]. □

2.6. **Adelic q-expansion.** Let $f \in M_k(\mathfrak{n}; R)$. By Proposition 2.3, f is determined by its q-expansions $f_{\mathfrak{c}} \in M_\infty(\mathfrak{c}; R)$ at $\infty(\mathfrak{c})$, where $\mathfrak{c}$ runs over the fixed set of representatives of $\mathcal{C}\ell_F^+$. We now define the adelic q-expansion of f, following [34, §2] (see also [23]). Let

$$M(R) := R[\mathcal{C}\ell_F^+] \bigoplus \left\{ \sum_{\mathfrak{b} \in \mathcal{I}} a(\mathfrak{b}) q^{\mathfrak{b}} \mid a(\mathfrak{b}) \in R \right\},$$

where $\mathcal{I}$ is the group of fractional ideals of F. For any fractional ideal $\mathfrak{c}$, consider the adelic expansion map

$$\psi_{\mathfrak{c}} : M_\infty(\mathfrak{c}; R) \to M(R)$$

sending $\sum_{\xi \in \mathfrak{c}_+ \cup \{0\}} a_\xi q^\xi$ to the element of $\sum_{\mathfrak{b} \in \mathcal{I} \cup \{(0)\}} a(\mathfrak{b}) q^{\mathfrak{b}} \in M(R)$ such that

$$a(\mathfrak{b}) = \begin{cases} a_\xi & \text{if } \mathfrak{b}\mathfrak{c} = (\xi) \text{ for some } \xi \in \mathfrak{c}_+, \\ a_0[\mathfrak{c}] & \text{if } \mathfrak{b} = (0), \\ 0 & \text{otherwise,} \end{cases}$$

where the first alternative means that $\mathfrak{b}\mathfrak{c}$ is the trivial element in $\mathcal{C}\ell_F^+$ so that $\mathfrak{b}\mathfrak{c}$ is principal with some totally positive generator ξ, uniquely determined up to totally positive units. The invariance property of $M_\infty(\mathfrak{c}; R)$ implies that this map does not depend on the choice of the totally positive generator ξ of the ideal $\mathfrak{b}\mathfrak{c}$. This hints to us why taking E-invariants is necessary in order to have a good theory of Hecke operators on q-expansions.

Let $\xi \in F_+^\times$ so that $\mathfrak{c}$ and $\mathfrak{c}' = \xi\mathfrak{c}$ are in the same narrow ideal class. Then

$$\psi_\xi : M_\infty(\mathfrak{c}; R) \to M_\infty(\mathfrak{c}'; R), \qquad \sum_{\xi' \in \mathfrak{c}_+ \cup \{0\}} a_{\xi'} q^{\xi'} \mapsto \sum_{\xi' \in \mathfrak{c}_+ \cup \{0\}} a_{\xi'} q^{\xi\xi'}$$

is an isomorphism of R-algebras and we have $\psi_{\mathfrak{c}'} \circ \psi_\xi = \psi_{\mathfrak{c}}$.

Adding up the maps $\psi_{\mathfrak{c}}$, gives a well-defined map, the adelic q-expansion:

$$\psi : \bigoplus_{\mathfrak{c} \in \mathcal{C}\ell_F^+} M_\infty(\mathfrak{c}; R) \to M(R),$$

where $\mathfrak{c}$ runs over the fixed set of representatives of the narrow ideal class group. The map ψ is independent of the choice of these representatives in the above sense.

Note that while any $f \in M_k(\mathfrak{n}; R)$ is determined by its adelic q-expansion, the latter does not have a direct geometric interpretation if $F \neq \mathbb{Q}$.

Let $\mathfrak{c}$ be a fractional ideal and $\mathfrak{q}$ be an integral ideal. It is immediate to check that the inclusion $\iota_{\mathfrak{q}} : M_\infty(\mathfrak{c}\mathfrak{q}; R) \hookrightarrow M_\infty(\mathfrak{c}; R)$ corresponds to the $\mathfrak{q}$-power map on adelic q-expansions, *i.e.*, $a(\mathfrak{b}, f) = a(\mathfrak{b}\mathfrak{q}, \iota_{\mathfrak{q}}(f))$ for all integral ideals $\mathfrak{b}$. Furthermore, the normalised trace map $t_{\mathfrak{q}} : M_\infty(\mathfrak{c}; R) \to M_\infty(\mathfrak{c}\mathfrak{q}; R)$ given by keeping only those terms which are indexed by elements of $\mathfrak{c}\mathfrak{q}$, corresponds to the $1/\mathfrak{q}$-power map on adelic q-expansions, *i.e.*, $a(\mathfrak{b}\mathfrak{q}, f) = a(\mathfrak{b}, t_{\mathfrak{q}}(f))$ for all integral ideals $\mathfrak{b}$.

3. Hecke operators

In this section we provide the operators on geometric Hilbert modular forms that are needed to prove the main theorem. In particular, we recall the constructions of the diamond operators and of Hecke operators of index coprime to the level and the characteristic. The core of this section is the definition of the Hecke operator $T_{\mathfrak{p}}$ for $\mathfrak{p} \mid p$. It is also the main input in our definition of Frobenius operators.

3.1. **Diamond operators.** Let $\mathfrak{q}$ be any ideal of $\mathfrak{o}$ relatively prime to $\mathfrak{n}$. Recall that by definition one has an exact sequence

$$0 \to (A \otimes_{\mathfrak{o}} \mathfrak{q})[\mathfrak{q}] \to A \otimes_{\mathfrak{o}} \mathfrak{q} \xrightarrow{i} A \to 0,$$

yielding, since by [26, §1.9] the Cartier dual of $A^\vee[\mathfrak{q}]$ is $(A \otimes_{\mathfrak{o}} \mathfrak{q})[\mathfrak{q}]$,

$$0 \to A^\vee[\mathfrak{q}] \to A^\vee \xrightarrow{i^\vee} (A \otimes_{\mathfrak{o}} \mathfrak{q})^\vee \to 0.$$

It results from these exact sequences that there is a canonical isomorphism

$$A^\vee \otimes_{\mathfrak{o}} \mathfrak{q}^{-1} \xrightarrow{\sim} (A \otimes_{\mathfrak{o}} \mathfrak{q})^\vee \tag{10}$$

Consider the automorphism $\langle \mathfrak{q} \rangle$ of $X_1^1(\mathfrak{n})$ sending (A, λ, μ) to $(A \otimes_{\mathfrak{o}} \mathfrak{q}, \lambda', \mu')$, where μ' is the composite of μ with the inverse of the canonical isomorphism $(A \otimes_{\mathfrak{o}} \mathfrak{q})[\mathfrak{n}] \simeq A[\mathfrak{n}]$ induced by $A \otimes_{\mathfrak{o}} \mathfrak{q} \to A$, and λ' is the following $\mathfrak{c}\mathfrak{q}^{-2}$-polarisation on $A \otimes_{\mathfrak{o}} \mathfrak{q}$:

$$\lambda' : (A \otimes_{\mathfrak{o}} \mathfrak{q}) \otimes_{\mathfrak{o}} \mathfrak{c}\mathfrak{q}^{-2} \underset{\sim}{\xrightarrow{\lambda \otimes \mathfrak{q}^{-1}}} A^\vee \otimes_{\mathfrak{o}} \mathfrak{q}^{-1} \underset{\sim}{\xrightarrow{(10)}} (A \otimes_{\mathfrak{o}} \mathfrak{q})^\vee.$$

Since $\langle \mathfrak{q} \rangle$ commutes with the E-action, it induces an automorphism of $X_1(\mathfrak{n})$ extending uniquely to an automorphism of $\overline{X_1(\mathfrak{n})}$. For every $\mathbb{Z}[\frac{1}{\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{n})}]$-algebra R the resulting R-linear endomorphism $\langle \mathfrak{q} \rangle$ of $M_k(\mathfrak{n}; R)$ sending $M_k(\mathfrak{c}\mathfrak{q}^{-1}, \mathfrak{n}; R)$ to $M_k(\mathfrak{c}\mathfrak{q}, \mathfrak{n}; R)$ is called the diamond operator (note that this operator is the inverse of the one defined by Hida in [23, 4.1.9]).

Lemma 3.1. *The morphism $\langle \mathfrak{q} \rangle$ sends the cusp $\infty(\mathfrak{c}\mathfrak{q}, \mathfrak{q}^{-1})$ to $\infty(\mathfrak{c}\mathfrak{q}^{-1})$ and identifies the formal completed rings at those cusps. In particular, the geometric q-expansion of any $f \in M_k(\mathfrak{c}\mathfrak{q}, \mathfrak{n}; R)$ at $\infty(\mathfrak{c}\mathfrak{q}, \mathfrak{q}^{-1})$ equals $(\langle \mathfrak{q}^{-1} \rangle f)_{\mathfrak{c}\mathfrak{q}^{-1}}$.*

Proof. The Tate object at $\bar{\phi}^{-1}(\infty(\mathfrak{c}\mathfrak{q}^{-1}))$ is $A = (\mathbb{G}_m \otimes_{\mathfrak{o}} \mathfrak{c}^{-1}\mathfrak{d}^{-1}\mathfrak{q})/q^{\mathfrak{o}}$ and its image under $\langle \mathfrak{q}^{-1} \rangle$ equals $A \otimes_{\mathfrak{o}} \mathfrak{q}^{-1} \simeq A/A[\mathfrak{q}] \simeq (\mathbb{G}_m \otimes_{\mathfrak{o}} \mathfrak{c}^{-1}\mathfrak{d}^{-1})/q^{\mathfrak{q}^{-1}}$, hence $\langle \mathfrak{q}^{-1} \rangle$ sends $\infty(\mathfrak{c}\mathfrak{q}^{-1})$ to $\infty(\mathfrak{c}\mathfrak{q}, \mathfrak{q}^{-1})$. Note that the formal completed rings at both cusps are given by $M_\infty(\mathfrak{c}\mathfrak{q}^{-1}, R)$, hence the claim. $\square$

It can be checked that the diamond operators induce an action of the ray class group of modulus $\mathfrak{n}$ on $M_k(\mathfrak{n}; R)$. Given any R-valued finite order Hecke character ε of F of conductor dividing $\mathfrak{n}$, we let $M_k(\mathfrak{n}, \varepsilon; R)$ denote the sub-R-module of $M_k(\mathfrak{n}; R)$ of forms on which $\langle \mathfrak{q} \rangle$ acts as $\varepsilon(\mathfrak{q})$ for all ideals $\mathfrak{q}$ relatively prime to $\mathfrak{n}$.

3.2. **Hecke operators away from the characteristic.** Let $\mathfrak{q}$ be a prime of F which is relatively prime to $\mathfrak{n}$, and let R be a $\mathbb{Z}[\frac{1}{\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{n}\mathfrak{q})}]$-algebra. The assumption that $\mathfrak{q}$ be invertible in R is removed in the next section when we construct the operators $T_{\mathfrak{p}}$ for $\mathfrak{p} \mid p$ in characteristic p.

Let $X = X_1(\mathfrak{n}) \times \mathrm{Spec}(R)$ and denote by $\overline{X}$ its minimal compactification, which is a normal projective scheme over R. One can define as in §2.1 a Hilbert modular variety Y of level $\Gamma_1(\mathfrak{n}) \cap \Gamma_0(\mathfrak{q})$ over R endowed with two finite surjective morphisms $\pi_1, \pi_2 : Y \rightrightarrows X$, corresponding to the forgetful and the 'quotient' maps on the level of moduli spaces, respectively. A careful inspection of the moduli theoretic definition of π_1 and π_2 (see [26, §1.10]) shows that they factor through one another. Therefore their extensions to the minimal compactifications fit in a commutative diagram:

$$\begin{tikzcd} \overline{Y} \arrow[rr, "\iota", "\sim"'] \arrow[dr, "\pi_1"'] & & \overline{Y} \arrow[dl, "\pi_2"] \\ & \overline{X} & \end{tikzcd} , \tag{11}$$

where ι is an automorphism, but not necessarily an involution, of $\overline{Y}$.

The Hecke operator $T_{\mathfrak{q}}^{(k)\vee}$ considered in [11, §2.4] is the R-linear endomorphism of $M_k(\mathfrak{n}; R)$ defined as the composite of the following three maps:

(i) the inclusion coming from the adjunction morphism:
$$\mathrm{H}^0(\overline{X}, \underline{\overline{\omega}}^{\otimes k}) \to \mathrm{H}^0(\overline{X}, \pi_{2*}\pi_2^*\underline{\overline{\omega}}^{\otimes k}) = \mathrm{H}^0(\overline{Y}, \pi_2^*\underline{\overline{\omega}}^{\otimes k});$$

(ii) the morphism
$$\mathrm{H}^0(\overline{Y}, \pi_2^*\underline{\overline{\omega}}^{\otimes k}) \to \mathrm{H}^0(\overline{Y}, \pi_1^*\underline{\overline{\omega}}^{\otimes k}) \tag{12}$$
induced by a morphisms of sheaves $\pi_2^*\underline{\overline{\omega}}^{\otimes k} \to \pi_1^*\underline{\overline{\omega}}^{\otimes k}$ (see [26, (1.11.1)]);

(iii) the morphism $\mathrm{H}^0(\overline{Y}, \pi_1^*\underline{\overline{\omega}}^{\otimes k}) = \mathrm{H}^0(\overline{X}, \pi_{1*}\pi_1^*\underline{\overline{\omega}}^{\otimes k}) \to \mathrm{H}^0(\overline{X}, \underline{\overline{\omega}}^{\otimes k})$ induced by the trace map $\pi_{1*}\pi_1^*\underline{\overline{\omega}}^{\otimes k} \to \underline{\overline{\omega}}^{\otimes k}$ relative to the finite morphism π_1, followed by the multiplication by $\mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})^{-1} \in R$.

Henceforth we set $T_{\mathfrak{q}}^{(k)} = T_{\mathfrak{q}}^{(k)\vee} \circ \langle \mathfrak{q} \rangle$ (this is the normalisation used by Gross [22]).

When R has characteristic 0, comparison with the complex analytic theory of automorphic forms for $\mathrm{GL}_2(F)$ implies that for all $f \in M_k(\mathfrak{n}; R)$ and for all non-zero ideals $\mathfrak{r} \subset \mathfrak{o}$ one has (see [34]):

$$a(\mathfrak{r}, T_{\mathfrak{q}}^{(k)} f) = a(\mathfrak{q}\mathfrak{r}, f) + \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{q})^{k-1} a(\mathfrak{r}/\mathfrak{q}, \langle \mathfrak{q} \rangle f) \tag{13}$$

where as usual $a(\mathfrak{r}/\mathfrak{q}, f) = 0$ if $\mathfrak{q} \nmid \mathfrak{r}$.

Together with the q-expansion principle, the above formula implies that the $T_{\mathfrak{q}}^{(k)}$s commute with each other. Moreover if R is a characteristic 0 field then $T_{\mathfrak{q}}^{(k)}$ is semi-simple.

Formula (13) remains valid when R has positive characteristic (relatively prime to $\mathfrak{q}$), which can also be seen by (a simpler version of) the argument given in the

next section. For any prime $\mathfrak{q}$ which is relatively prime to $\mathfrak{n}$ and to the characteristic of R and for any $f \in M_k(\mathfrak{n}, \varepsilon; R)$, where ε is a R-valued finite order Hecke character of F of conductor dividing $\mathfrak{n}$, we have:

$$a(\mathfrak{r}, T_{\mathfrak{q}}^{(k)} f) = a(\mathfrak{q}\mathfrak{r}, f) + \varepsilon(\mathfrak{q})\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{k-1} a(\mathfrak{r}/\mathfrak{q}, f). \tag{14}$$

Since Eisenstein series with central character ε generate a supplement to the space of cusp forms in $M_k(\mathfrak{n}, \varepsilon; R)$, and since the only Eisenstein series having non-zero constant term at the ∞ cusps are those attached to a pair of characters φ and $\varepsilon\varphi^{-1}$ with φ a character of $\mathcal{C}\ell_F^+$, one has the following formula for the constant term (see [8, §2.2])

$$a((0), T_{\mathfrak{q}}^{(k)} f) = a((0), f)[\mathfrak{q}] + \varepsilon(\mathfrak{q})\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{q})^{k-1} a((0), f)[\mathfrak{q}^{-1}]. \tag{15}$$

3.3. **Hecke operators in characteristic p.** Fix a prime number p which is relatively prime to $\mathfrak{n}$ and fix a prime $\mathfrak{p}$ dividing p. In §3.2 we defined a $\mathbb{Q}_p$-linear endomorphism $T_{\mathfrak{p}}^{(k)}$ of $M_k(\mathfrak{n}; \mathbb{Q}_p)$. It follows from (13) and the q-expansion principle that $T_{\mathfrak{p}}^{(k)}$ induces a $\mathbb{Z}_p$-linear endomorphism of $M_k(\mathfrak{n}; \mathbb{Z}_p)$, hence a $\mathbb{F}_p$-linear endomorphism of $M_k(\mathfrak{n}; \mathbb{Z}_p) \otimes \mathbb{F}_p$. The aim of this section is to extend $T_{\mathfrak{p}}^{(k)}$ to an endomorphism of the whole space $M_k(\mathfrak{n}; \mathbb{F}_p)$ for all $k \geq 1$. Note that such an endomorphism would be uniquely determined by its action on q-expansions.

Emerton, Reduzzi and Xiao propose in [17, §3] a different approach in the derived category using the dualising trace map.

An original feature in our approach is the detailed study of the formal skeleton of the $T_{\mathfrak{p}}$-correspondence, allowing us to compute the effect of $T_{\mathfrak{p}}^{(k)}$ on q-expansions of modular forms. This is based on an isomorphism between the (minimal compactification of the) Hilbert modular variety $\overline{Y(\mathfrak{c})}$ with $\Gamma_0(\mathfrak{p})$-level structure and a suitable relative normalisation (see Lemma 3.2) allowing us to determine its formal completion at the cusps $\infty_{\mathfrak{c}}$ and $0_{\mathfrak{c}}$ in Corollary 3.4.

3.3.1. *Integral models by normalisation.* For any fractional ideal $\mathfrak{c}$, let $X(\mathfrak{c}) = X_1(\mathfrak{c}, \mathfrak{n}) \times \mathrm{Spec}(\mathbb{Z}_p)$. Its minimal compactification $\overline{X(\mathfrak{c})}$ is a flat projective scheme over $\mathbb{Z}_p$, the fibres of which are normal and geometrically irreducible.

The scheme $Y(\mathfrak{c})_{\mathbb{Q}_p}$ considered in the previous section for $R = \mathbb{Q}_p$ admits a model $Y(\mathfrak{c})$ over $\mathbb{Z}_p$ which is normal and is a relative complete intersection. Namely one can define $Y(\mathfrak{c})$ as the quotient by E of the moduli space with $\Gamma_0(\mathfrak{p})$-level structure $Y^1(\mathfrak{c})$ defined by Stamm and Pappas [29]. Its minimal compactification $\overline{Y(\mathfrak{c})}$ is proper over $\mathbb{Z}_p$ and normal.

By construction (see [26, §1.9]) there exist degeneracy morphisms

$$\pi_1 : \overline{Y(\mathfrak{c})} \to \overline{X(\mathfrak{c})} \text{ and } \pi_2 : \overline{Y(\mathfrak{c})} \to \overline{X(\mathfrak{c}\mathfrak{p})},$$

which are proper (see [35, Tag 01W6]), surjective and generically finite, but may not be finite over the non-ordinary locus in the special fibre (see [21]). In particular, since all schemes are separated, the morphism $(\pi_1, \pi_2) : \overline{Y(\mathfrak{c})} \to \overline{X(\mathfrak{c})} \times \overline{X(\mathfrak{c}\mathfrak{p})}$ is proper, too.

Define $\overline{Z(\mathfrak{c})'}$ (resp. $\overline{Z(\mathfrak{c})''}$) as the relative normalisation of

$$\overline{Y(\mathfrak{c})}_{\mathbb{Q}_p} \xrightarrow{\pi_1} \overline{X(\mathfrak{c})}_{\mathbb{Q}_p} \to \overline{X(\mathfrak{c})} \ \big(\text{resp. } \overline{Y(\mathfrak{c})}_{\mathbb{Q}_p} \xrightarrow{\pi_2} \overline{X(\mathfrak{cp})}_{\mathbb{Q}_p} \to \overline{X(\mathfrak{cp})}\ \big).$$

Note that while the morphism $\pi_1 : \overline{Z(\mathfrak{c})'} \to \overline{X(\mathfrak{c})}$ is finite, one cannot in general define a morphism $\overline{Z(\mathfrak{c})'} \to \overline{X(\mathfrak{cp})}$ extending $\pi_2 : \overline{Y(\mathfrak{c})}_{\mathbb{Q}_p} \to \overline{X(\mathfrak{cp})}_{\mathbb{Q}_p}$. To palliate this problem we consider the relative normalisation $\overline{Z(\mathfrak{c})}$ of

$$\overline{Y(\mathfrak{c})}_{\mathbb{Q}_p} \xrightarrow{(\pi_1,\pi_2)} \overline{X(\mathfrak{c})}_{\mathbb{Q}_p} \times \overline{X(\mathfrak{cp})}_{\mathbb{Q}_p} \to \overline{X(\mathfrak{c})} \times \overline{X(\mathfrak{cp})}. \tag{16}$$

By functoriality of relative normalisation (e.g. [35, Tag 035J]) there exist canonical commutative diagrams:

$$\begin{tikzcd} \overline{Z(\mathfrak{c})} \arrow[r, "\nu_1"] \arrow[dr, "\pi_1'''"'] & \overline{Z(\mathfrak{c})'} \arrow[d, "\pi_1'"] \\ & \overline{X(\mathfrak{c})} \end{tikzcd}, \text{ and } \begin{tikzcd} \overline{Z(\mathfrak{c})} \arrow[r, "\nu_2"] \arrow[dr, "\pi_2'''"'] & \overline{Z(\mathfrak{c})''} \arrow[d, "\pi_2''"] \\ & \overline{X(\mathfrak{cp})} \end{tikzcd},$$

where the horizontal morphisms are proper (e.g. [35, Tag 01W6]) and birational, as they induce isomorphisms on dense $\mathbb{Q}_p$-fibres. Since the schemes $\overline{Z(\mathfrak{c})}$, $\overline{Z(\mathfrak{c})'}$ and $\overline{Z(\mathfrak{c})''}$ are normal (e.g. [35, Tag 035L]), it follows from [35, Tag 0AB1]) that ν_i becomes an isomorphism when restricted to any open over which π_i''' is finite, in particular over the ordinary locus (see [17, Proposition 3.7]).

Finally we prove a useful lemma by reproducing an argument communicated to us by Kai-Wen Lan.

Lemma 3.2. *The morphism $(\pi_1, \pi_2) : \overline{Y(\mathfrak{c})} \to \overline{X(\mathfrak{c})} \times \overline{X(\mathfrak{cp})}$ is finite and there is a canonical isomorphism between $\overline{Z(\mathfrak{c})}$ and $\overline{Y(\mathfrak{c})}$.*

Proof. Suppose first that (π_1, π_2) is integral. Then by the universal property of normalisation (e.g. [35, Tag 035I]) there exists a unique morphism ν inducing identity on dense $\mathbb{Q}_p$-fibres and fitting in the following commutative diagram

$$\begin{tikzcd} \overline{Z(\mathfrak{c})} \arrow[r, "\nu"] \arrow[dr, "(\pi_1''',\pi_2''')"'] & \overline{Y(\mathfrak{c})} \arrow[d, "(\pi_1,\pi_2)"] \\ & \overline{X(\mathfrak{c})} \times \overline{X(\mathfrak{cp})}. \end{tikzcd}$$

Since (π_1''', π_2''') is finite by construction (e.g. [35, Tag 01WJ]), one deduces that ν is finite, too (e.g. [35, Tag 035D]), which together with the fact that $\overline{Y(\mathfrak{c})}$ is normal, implies that ν is an isomorphism (see [35, Tag 0AB1]).

To prove that (π_1, π_2) is finite, by properness it suffices to show that it is quasi-finite, for which one can restrict to the non-cuspidal locus (since the cuspidal locus consists of points). We will establish the finiteness of the morphism $(\pi_1, \pi_2) : Y^1(\mathfrak{c}) \to X^1(\mathfrak{c}) \times X^1(\mathfrak{cp})$ on the level of fine moduli spaces, since taking the quotient by the finite group E preserves this property.

Let $Z^1(\mathfrak{c})$ be the relative normalisation of

$$Y^1(\mathfrak{c})_{\mathbb{Q}_p} \overset{(\pi_1,\pi_2)}{\longrightarrow} X^1(\mathfrak{c})_{\mathbb{Q}_p} \times X^1(\mathfrak{c}\mathfrak{p})_{\mathbb{Q}_p} \to X^1(\mathfrak{c}) \times X^1(\mathfrak{c}\mathfrak{p}) \tag{17}$$

and consider the pullbacks $(\pi_1''')^*\mathcal{A}(\mathfrak{c})$ and $(\pi_2''')^*\mathcal{A}(\mathfrak{c}\mathfrak{p})$ of the universal abelian schemes over $X^1(\mathfrak{c})$ and $X^1(\mathfrak{c}\mathfrak{p})$ defined in section 2.1. Since $Z^1(\mathfrak{c})$ is noetherian and normal, Raynaud's theorem (see [32, Cor. IX.1.4] or [18, Prop. I.2.7]) implies that the universal isogeny between the $\mathbb{Q}_p$-fibres of these two pullbacks extends, yielding by universal property of the fine moduli space $Y^1(\mathfrak{c})$ a commutative diagram

$$\begin{tikzcd} Z^1(\mathfrak{c}) \arrow[r, "\nu"] \arrow[dr, "(\pi_1''',\pi_2''')"'] & Y^1(\mathfrak{c}) \arrow[d, "(\pi_1,\pi_2)"] \\ & X^1(\mathfrak{c}) \times X^1(\mathfrak{c}\mathfrak{p}). \end{tikzcd}$$

By the same arguments as above, the finiteness of (π_1''',π_2''') and normality of $Y^1(\mathfrak{c})$ imply that ν is an isomorphism and that (π_1,π_2) is finite, finishing the proof of the lemma. $\square$

3.3.2. *The formal skeleton of the $T_\mathfrak{p}$ correspondence.* As the $T_\mathfrak{p}$-correspondence does not preserve individual components, we consider $X = \coprod_{\mathfrak{c}\in\mathcal{C}\ell_F^+} X(\mathfrak{c})$ and $Y = \coprod_{\mathfrak{c}\in\mathcal{C}\ell_F^+} Y(\mathfrak{c})$. Since we will be mostly interested in the effect of $T_\mathfrak{p}$ on q-expansions, it is natural to study the pull-back of π_1 and π_2 to formal neighbourhoods of the infinity cusps of $\overline{X}$. Since those cusps belong to the ordinary locus, by the discussion in the previous paragraph one can use the definition $\overline{Y}$ based on normalisations.

Proposition 3.3. *The inverse image under π_1 of the cusp $\infty(\mathfrak{c}) : \mathrm{Spec}(\mathbb{Z}_p) \to \overline{X(\mathfrak{c})}$ consists of two cusps denoted $\infty_\mathfrak{c}, 0_\mathfrak{c} : \mathrm{Spec}(\mathbb{Z}_p) \to \overline{Y(\mathfrak{c})}$, and the formal completion of $\overline{Y(\mathfrak{c})}$ along $\infty_\mathfrak{c}$ (resp. $0_\mathfrak{c}$) is given by*

$$\mathcal{Y}_\infty(\mathfrak{c}) = \mathrm{Spf}(M_\infty(\mathfrak{c};\mathbb{Z}_p)) \ \textit{(resp. } \mathcal{Y}_0(\mathfrak{c}) = \mathrm{Spf}(M_\infty(\mathfrak{c}\mathfrak{p}^{-1};\mathbb{Z}_p))).$$

Proof. By [13, Theorem 8.6], the formal completion $\mathcal{X}(\mathfrak{c})$ of $\overline{X(\mathfrak{c})}$ at $\infty(\mathfrak{c})$ is given by $\mathrm{Spf}(M_\infty(\mathfrak{c};\mathbb{Z}_p))$. Also by *loc. cit.* the local completed ring of $\overline{Y(\mathfrak{c})}_{\mathbb{Q}_p}$ at $\infty_\mathfrak{c}$ (resp. $0_\mathfrak{c}$) is given by $M_\infty(\mathfrak{c};\mathbb{Q}_p)$ (resp. $M_\infty(\mathfrak{c}\mathfrak{p}^{-1};\mathbb{Q}_p)$).

The scheme $\overline{X(\mathfrak{c})}$ is of finite type over $\mathbb{Z}_p$, hence excellent by [16, IV.7.8.6]. Since by [16, IV.7.8.3(vii)] normalisation and completion commute for reduced excellent local rings, it follows from the definition of $\overline{Z(\mathfrak{c})}$ that its local completion at $\pi_1^{-1}(\infty(\mathfrak{c}))$ is given by the normalisation of $M_\infty(\mathfrak{c};\mathbb{Z}_p)$ in $M_\infty(\mathfrak{c};\mathbb{Q}_p)\times M_\infty(\mathfrak{c}\mathfrak{p}^{-1};\mathbb{Q}_p)$. In particular, it follows that $\pi_1^{-1}(\infty(\mathfrak{c}))$ consists of two points.

Since for any $\mathfrak{c}' \supset \mathfrak{c}$ the ring $M_\infty(\mathfrak{c}';\mathbb{Z}_p)$ is a finitely generated module and hence integral over $M_\infty(\mathfrak{c};\mathbb{Z}_p)$, it remains to show that any element of $M_\infty(\mathfrak{c}';\mathbb{Q}_p)$ which is integral over $M_\infty(\mathfrak{c};\mathbb{Z}_p)$, belongs to $M_\infty(\mathfrak{c}';\mathbb{Z}_p)$.

Since $\mathcal{X}(\mathfrak{c}')$ is normal and excellent its completed local ring at infinity $M_\infty(\mathfrak{c}';\mathbb{Z}_p)$ is a normal ring, hence it is integrally closed in $M_\infty(\mathfrak{c}';\mathbb{Q}_p)$. Since the embedding

$M_\infty(\mathfrak{c};\mathbb{Z}_p) \to M_\infty(\mathfrak{c}';\mathbb{Q}_p)$ factors through $M_\infty(\mathfrak{c}';\mathbb{Z}_p)$, any element of $M_\infty(\mathfrak{c}';\mathbb{Q}_p)$ which is integral over $M_\infty(\mathfrak{c};\mathbb{Z}_p)$ is in particular integral over $M_\infty(\mathfrak{c}';\mathbb{Z}_p)$, hence belongs to the latter ring, as was to be shown.

Another way to see this is to use that $M_\infty(\mathfrak{c};\mathbb{Z}_p)$ is isomorphic to the $\mathfrak{o}_+^\times$-invariants in an intersection of power series rings $R_\sigma^\wedge(\mathfrak{c};\mathbb{Z}_p)$ as in [13, §2], where σ runs over a rational cone decomposition of $F_+^\times$. The claim then follows from the well-known analogous statement for polynomial and power series rings. $\square$

By looking at the generic fibre, one can check that π_2 sends $0_\mathfrak{c}$ (resp. $\infty_\mathfrak{c}$) to $\infty(\mathfrak{cp},\mathfrak{p}^{-1})$ (resp. $\infty(\mathfrak{cp})$). The description of π_1 in Proposition 3.3, along with an analogous analysis of π_2 gives the following result.

Corollary 3.4. *(a) The formal completion of π_1 along $\infty_\mathfrak{c}$ gives an isomorphism $\mathcal{Y}_\infty(\mathfrak{c}) \to \mathcal{X}_\infty(\mathfrak{c})$, which corresponds to the identity on $M_\infty(\mathfrak{c};\mathbb{Z}_p)$.*

(b) The formal completion of π_1 along $0_\mathfrak{c}$ gives a morphism $\mathcal{Y}_0(\mathfrak{c}) \to \mathcal{X}_\infty(\mathfrak{c})$, which corresponds to the natural inclusion $M_\infty(\mathfrak{c};\mathbb{Z}_p) \hookrightarrow M_\infty(\mathfrak{cp}^{-1};\mathbb{Z}_p)$.

(c) The formal completion of π_2 along $\infty_\mathfrak{c}$ gives a morphism $\mathcal{Y}_\infty(\mathfrak{c}) \to \mathcal{X}_\infty(\mathfrak{cp})$, which corresponds to the natural inclusion $M_\infty(\mathfrak{cp};\mathbb{Z}_p) \hookrightarrow M_\infty(\mathfrak{c};\mathbb{Z}_p)$.

(d) The formal completion of π_2 along $0_\mathfrak{c}$ gives an isomorphism $\mathcal{Y}_0(\mathfrak{c}) \to \mathcal{X}_\infty(\mathfrak{cp},\mathfrak{p}^{-1})$, which corresponds to the identity on $M_\infty(\mathfrak{cp}^{-1};\mathbb{Z}_p)$.

Note that in view of [16, IV.7.8.3(vii)], Proposition 3.3 implies that $\overline{Y(\mathfrak{c})}_{\mathbb{F}_p}$ has (at most) two horizontal (in the sense of [21]) geometrically irreducible components, one containing $0_{\mathfrak{c},\mathbb{F}_p}$ and the other containing $\infty_{\mathfrak{c},\mathbb{F}_p}$.

3.3.3. *Integrality of the normalised trace.* Let $U \to \mathrm{Spec}(\mathbb{Z}_p)$ be any open affine in $\overline{X(\mathfrak{c})}$ containing $\infty(\mathfrak{c})$. By §2.5 the local completed ring of U at $\infty(\mathfrak{c})$ is given by $M_\infty(\mathfrak{c};\mathbb{Z}_p)$.

The sheaves $\pi_i^*\underline{\omega}^{\otimes k}$ $(i=1,2)$ can be trivialised over $\mathcal{Y}_\infty(\mathfrak{c}) \coprod \mathcal{Y}_0(\mathfrak{c})$ yielding the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^0(\pi_1^{-1}(U),\pi_2^*\underline{\omega}^{\otimes k}) \arrow[r] \arrow[d] & M_\infty(\mathfrak{c};\mathbb{Z}_p) \arrow[d, hook] & \times & M_\infty(\mathfrak{cp}^{-1};\mathbb{Z}_p) \arrow[d, hook] \\
\mathrm{H}^0(\pi_1^{-1}(U_{\mathbb{Q}_p}),\pi_2^*\underline{\omega}^{\otimes k}) \arrow[r, hook] \arrow[d] & M_\infty(\mathfrak{c};\mathbb{Q}_p) \arrow[d, "\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^k"'] & \times & M_\infty(\mathfrak{cp}^{-1};\mathbb{Q}_p) \arrow[d, equal] \\
\mathrm{H}^0(\pi_1^{-1}(U_{\mathbb{Q}_p}),\pi_1^*\underline{\omega}^{\otimes k}) \arrow[r, hook] \arrow[d, "\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-1}\mathrm{Tr}(\pi_1)"'] & M_\infty(\mathfrak{c};\mathbb{Q}_p) \arrow[d, "\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{-1}"'] & \times & M_\infty(\mathfrak{cp}^{-1};\mathbb{Q}_p) \arrow[dll, "t_\mathfrak{p}"] \\
\mathrm{H}^0(U_{\mathbb{Q}_p},\underline{\omega}^{\otimes k}) \arrow[r, hook] & M_\infty(\mathfrak{c};\mathbb{Q}_p) & + &
\end{tikzcd} \tag{18}$$

where all horizontal morphisms are q-expansions at $\pi_1^{-1}(\infty(\mathfrak{c})) = \{\infty_\mathfrak{c}, 0_\mathfrak{c}\}$ (see Corollary 3.4), except the last one which is the usual q-expansions map (9) at $\infty(\mathfrak{c})$. The middle vertical arrow comes from the connecting morphism of sheaves

$\pi_2^*\underline{\omega}^{\otimes k} \to \pi_1^*\underline{\omega}^{\otimes k}$ described in [26, §1.11.1] and $t_{\mathfrak{p}} : M_\infty(\mathfrak{c}\mathfrak{p}^{-1};\mathbb{Q}_p) \to M_\infty(\mathfrak{c};\mathbb{Q}_p)$ denotes the normalised trace (see §2.6). The commutativity of the middle and lower square follow form well known computations over $\mathbb{Q}_p$. The commutativity of the upper square comes from the commutativity of q-expansion and base change. Note that we do *not* claim the injectivity of the upper horizontal map, since when π_1 is not finite $\pi_1^{-1}(U_{\mathbb{F}_p})$ has geometrically irreducible components which contain neither $\infty_{\mathfrak{c},\mathbb{F}_p}$ nor $0_{\mathfrak{c},\mathbb{F}_p}$.

Denote by η the composite of the morphisms of the first column in (18). Following the vertical maps on the right side of (18) implies that if $k \geq 1$ then

$$\eta(\mathrm{H}^0(U, \pi_{1*}\pi_2^*\underline{\omega}^{\otimes k})) \subset \mathrm{H}^0(U_{\mathbb{Q}_p}, \underline{\omega}^{\otimes k}) \cap M_\infty(\mathfrak{c};\mathbb{Z}_p)$$

Proposition 2.3 implies that

$$\mathrm{H}^0(U_{\mathbb{Q}_p}, \underline{\omega}^{\otimes k}) \cap M_\infty(\mathfrak{c};\mathbb{Z}_p) = \mathrm{H}^0(U, \underline{\omega}^{\otimes k}),$$

and that for any $U' \subset U$ as above the corresponding η' and η agree.

Consider a finite open affine cover $(U_i)_{i\in I(\mathfrak{c})}$ of $\overline{X(\mathfrak{c})}$ such that $\infty(\mathfrak{c}) \in U_i$ for all $i \in I(\mathfrak{c})$ (one may, for example, consider an embedding of $\overline{X(\mathfrak{c})}$ in a projective space over $\mathbb{Z}_p$ and take complements of hyperplane sections which do not meet $\infty(\mathfrak{c})$). Let $I = \bigcup_{\mathfrak{c}\in\mathcal{C}\ell_F^+} I(\mathfrak{c})$ with $\mathfrak{c}$ running over the fixed set of representatives of $\mathcal{C}\ell_F^+$. Then $(U_i)_{i\in I}$ is an open affine cover of $\overline{X}$. By the above discussion we obtain:

Proposition 3.5. *For $k \geq 1$, diagram* (18) *defines a morphism of sheaves η : $\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k} \to \underline{\omega}^{\otimes k}$ on $\overline{X}$.*

3.3.4. *Definition of $T_{\mathfrak{p}}^{(k)}$.* The adjunction morphism $\underline{\omega}_{\mathbb{F}_p}^{\otimes k} \to \pi_{2*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}$ of sheaves on $\overline{X}_{\mathbb{F}_p}$ induces:

(19)

$$\mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \underline{\omega}_{\mathbb{F}_p}^{\otimes k}) \xrightarrow{\pi_2^*} \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \pi_{2*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) = \mathrm{H}^0(\overline{Y}_{\mathbb{F}_p}, \pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) = \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}).$$

By Proposition 3.5 one has a morphism:

$$\mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, (\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k}) \otimes \mathbb{F}_p) \xrightarrow{\eta\otimes\mathbb{F}_p} \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \underline{\omega}_{\mathbb{F}_p}^{\otimes k}), \tag{20}$$

which we will now extend to $\mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k})$. The short exact sequence

$$0 \to \pi_2^*\underline{\omega}^{\otimes k} \xrightarrow{\cdot p} \pi_2^*\underline{\omega}^{\otimes k} \to \pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k} \to 0$$

of sheaves on $\overline{X}$ yields the long exact sequence:

$$\begin{aligned} 0 \to \pi_{1*}\pi_2^*\underline{\omega}^{\otimes k} \xrightarrow{\cdot p} \pi_{1*}\pi_2^*\underline{\omega}^{\otimes k} \to \pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k} \to \\ \to \mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k} \xrightarrow{\cdot p} \mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k} \to \mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k} \end{aligned}$$

By [17, Proposition 3.18], the support of $\mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}$ has codimension at least 2 in $\overline{X}_{\mathbb{F}_p}$, *i.e.*, its localisation $(\mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k})_x$ at any codimension 1 point x in $\overline{X}_{\mathbb{F}_p}$

vanishes. Localising at any such x the above long exact sequence then yields

$$(\mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k})_x \otimes \mathbb{F}_p = ((\mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k}) \otimes \mathbb{F}_p)_x = 0.$$

Since π_1 is proper, the sheaf $\mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k}$ is coherent and Nakayama's lemma implies that $(\mathrm{R}^1\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k})_x = 0$. The above long exact sequence then implies that the morphism of sheaves

$$(\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k}) \otimes \mathbb{F}_p \to \pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k} \tag{21}$$

is an isomorphism outside a subscheme Z of $\overline{X}_{\mathbb{F}_p}$ of codimension at least 2. Proposition 3.5 then yields:

(22)

$$\mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) \xrightarrow{\mathrm{res}} \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}\backslash Z, \pi_{1*}\pi_2^*\underline{\omega}_{\mathbb{F}_p}^{\otimes k}) \underset{\sim}{\overset{(21)}{\longleftarrow}} \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}\backslash Z, (\pi_{1*}\pi_2^*\underline{\omega}^{\otimes k}) \otimes \mathbb{F}_p) \to$$
$$\xrightarrow{\eta\otimes\mathbb{F}_p} \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}\backslash Z, \underline{\omega}_{\mathbb{F}_p}^{\otimes k}) = \mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \underline{\omega}_{\mathbb{F}_p}^{\otimes k}),$$

where the last equality follows from the fact that $\overline{X}_{\mathbb{F}_p}$ is normal.

For $k \geq 1$, the endomorphism $T_{\mathfrak{p}}^{(k)\vee}$ of $\mathrm{H}^0(\overline{X}_{\mathbb{F}_p}, \underline{\omega}_{\mathbb{F}_p}^{\otimes k})$ is defined as the composite of (19) with (22). Finally we set:

$$T_{\mathfrak{p}}^{(k)} = T_{\mathfrak{p}}^{(k)\vee} \circ \langle \mathfrak{p} \rangle : M_k(\mathfrak{n}; \mathbb{F}_p) \to M_k(\mathfrak{n}; \mathbb{F}_p). \tag{23}$$

3.3.5. *The effect of $T_{\mathfrak{p}}$ on q-expansions.* We first compute the effect of (19) on q-expansions.

By Lemma 3.1 and Corollary 3.4 for any $f \in M_k(\mathfrak{n}; \mathbb{F}_p)$ the q-expansion of $\pi_2^*(f) \in \mathrm{H}^0(\overline{Y}_{\mathbb{F}_p}, \pi_2^*\underline{\omega}^{\otimes k})$ at $\pi_1^{-1}(\infty(\mathfrak{c})) = \{\infty_{\mathfrak{c}}, 0_{\mathfrak{c}}\}$ equals

$$(\iota_{\mathfrak{p}}(f_{\mathfrak{cp}}), (\langle \mathfrak{p}^{-1} \rangle f)_{\mathfrak{cp}^{-1}}) \in M_\infty(\mathfrak{c}; \mathbb{F}_p) \times M_\infty(\mathfrak{cp}^{-1}; \mathbb{F}_p), \tag{24}$$

where $\iota_{\mathfrak{p}} : M_\infty(\mathfrak{cp}; \mathbb{F}_p) \to M_\infty(\mathfrak{c}; \mathbb{F}_p)$ is the natural inclusion (see §2.6).

Combining (18) and (24) shows that for $f \in M_k(\mathfrak{n}; \mathbb{F}_p)$ one has:

$$(T_{\mathfrak{p}}^{(k)\vee} f)_{\mathfrak{c}} = \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{p})^{k-1} f_{\mathfrak{cp}} + t_{\mathfrak{p}}((\langle \mathfrak{p}^{-1} \rangle f)_{\mathfrak{cp}^{-1}}), \text{ and}$$

$$(T_{\mathfrak{p}}^{(k)} f)_{\mathfrak{c}} = t_{\mathfrak{p}}(f_{\mathfrak{cp}^{-1}}) + \mathrm{N}_{\mathrm{F}/\mathbb{Q}}(\mathfrak{p})^{k-1} (\langle \mathfrak{p} \rangle f)_{\mathfrak{cp}}, \tag{25}$$

where $t_{\mathfrak{p}} : M_\infty(\mathfrak{cp}^{-1}; \mathbb{F}_p) \to M_\infty(\mathfrak{c}; \mathbb{F}_p)$ denotes the normalised trace (see §2.6).

Letting $\mathfrak{c}$ run over the fixed set of representatives of $\mathcal{C}\ell_F^+$, the facts established in §2.6 lead to a proof of Theorem 1.2.

Note that by the q-expansion principle the Hecke and diamond operators generate a commutative algebra.

Due to its importance for the sequel, we draw the reader's attention to the fact that an inspection of the action on formal q-expansions shows that, in characteristic p, $T_{\mathfrak{p}}^{(k)}$ acts like a $U_{\mathfrak{p}}$-operator as soon as $k > 1$.

3.4. **The Hasse invariant.** In the sequel, the Hasse invariant $h \in M_{p-1}(\mathfrak{n}, 1; \overline{\mathbb{F}}_p)$ will play a fundamental role for passing from weight 1 to weight p; in combination with the Hecke operators $T_{\mathfrak{p}}$ for $\mathfrak{p}$ dividing p, the Frobenius operators will be derived from it in the next subsection. The Hasse invariant h has q-expansion equal to 1 at the cusp at ∞ of each connected component of $\overline{X_1(\mathfrak{n})}$.

For the existence of the Hasse invariant, the reader can either refer to [1, §7.12-14] or [26, §1.5]. By p. 740 of the latter reference, the Hasse invariant, which is *a priori* constructed on $\overline{X_1^1(\mathfrak{n})}$, is independent of the polarisation and thus E-invariant, so that it descends to $\overline{X_1(\mathfrak{n})}$, as needed.

3.5. **The Frobenius operators.** In this section we define operators V_P taking weight 1 forms to weight p forms. A main feature of our treatment is that its only ingredients are the Hecke operators $T_{\mathfrak{p}}^{(1)}$ and $T_{\mathfrak{p}}^{(p)}$ for primes $\mathfrak{p} \mid p$ and the total Hasse invariant.

Let ε be a fixed $\overline{\mathbb{F}}_p^{\times}$-valued Hecke character of F of conductor dividing $\mathfrak{n}$.

For squarefree ideals $P \subset \mathfrak{o}$ dividing p, we define the *Frobenius operator*

$$V_P : M_1(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p) \to M_p(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p) \tag{26}$$

inductively as follows:

$$V_1 := V_{\mathfrak{o}} := h \text{ and } V_{P\mathfrak{p}} := \varepsilon(\mathfrak{p})^{-1}\left(V_P T_{\mathfrak{p}}^{(1)} - T_{\mathfrak{p}}^{(p)} V_P\right), \tag{27}$$

where $\mathfrak{p}$ is a prime ideal dividing (p) coprime to P.

The description of the action on q-expansions in the following proposition shows that the V_P do not depend on the order in which the prime divisors of P are used in the recursive definition, hence they are well-defined. It also shows that V_P commutes with the Hecke operators $T_{\mathfrak{q}}$ for all $\mathfrak{q} \nmid p\mathfrak{n}$.

Proposition 3.6. *For every squarefree ideal $P \subset \mathfrak{o}$ dividing p one has:*

$$\begin{aligned} a((0), V_P f) &= a((0), f)[P^{-1}], \text{ and} \\ a(\mathfrak{r}, V_P f) &= a(\mathfrak{r}/P, f) \text{ for all } (0) \neq \mathfrak{r} \subset \mathfrak{o}. \end{aligned} \tag{28}$$

Proof. If $P = \mathfrak{o}$, the result is trivial. Assume that the result holds for P and let $\mathfrak{p}$ be a prime ideal dividing (p) coprime to P. Using (1), for all $(0) \neq \mathfrak{r} \subset \mathfrak{o}$ we have

$$\begin{aligned} a(\mathfrak{r}, V_{P\mathfrak{p}} f) &= \varepsilon(\mathfrak{p})^{-1}\left(a(\mathfrak{r}, V_P T_{\mathfrak{p}}^{(1)} f) - a(\mathfrak{r}, T_{\mathfrak{p}}^{(p)} V_P f)\right) \\ &= \varepsilon(\mathfrak{p})^{-1}\left(a(\mathfrak{r}/P, T_{\mathfrak{p}}^{(1)} f) - a(\mathfrak{r}\mathfrak{p}, V_P f)\right) \\ &= \varepsilon(\mathfrak{p})^{-1}\left(a(\mathfrak{r}\mathfrak{p}/P, f) + a(\mathfrak{r}/(P\mathfrak{p}), \langle\mathfrak{p}\rangle f) - a(\mathfrak{r}\mathfrak{p}/P, f)\right) \\ &= a(\mathfrak{r}/(P\mathfrak{p}), f), \text{while} \end{aligned}$$

$$\begin{aligned}
a((0),V_{P\mathfrak{p}}f) &= \varepsilon(\mathfrak{p})^{-1}\left(a((0),V_PT^{(1)}_{\mathfrak{p}}f) - a((0),T^{(p)}_{\mathfrak{p}}V_Pf)\right)\\
&= \varepsilon(\mathfrak{p})^{-1}\left(a((0),T^{(1)}_{\mathfrak{p}}f)[P^{-1}] - a((0),V_Pf)[\mathfrak{p}]\right)\\
&= \varepsilon(\mathfrak{p})^{-1}\left(a((0),f)[\mathfrak{p}P^{-1}] + a((0),\langle\mathfrak{p}\rangle f)[(P\mathfrak{p})^{-1}] - a((0),f)[\mathfrak{p}P^{-1}]\right)\\
&= a((0),f)[(P\mathfrak{p})^{-1}].
\end{aligned}$$

Note that the calculations make sense even if P does not divide $\mathfrak{r}$; it is here that the squarefreeness is used. □

Lemma 3.7. *Let $\mathfrak{p} \subset \mathfrak{o}$ be prime and $P \subset \mathfrak{o}$ be squarefree, both dividing p. Then we have*

$$T^{(p)}_{\mathfrak{p}}V_P = \begin{cases} V_{P/\mathfrak{p}} & \text{if } \mathfrak{p} \mid P,\\ V_PT^{(1)}_{\mathfrak{p}} - \varepsilon(\mathfrak{p})V_{P\mathfrak{p}} & \text{if } \mathfrak{p} \nmid P.\end{cases}$$

Proof. We first compute the adelic q-expansion of $T^{(p)}_{\mathfrak{p}}V_Pf$:

$$a(\mathfrak{r},T^{(p)}_{\mathfrak{p}}V_Pf) = a(\mathfrak{r}\mathfrak{p},V_Pf) = a(\mathfrak{r}\mathfrak{p}/P,f), \text{and}$$
$$a((0),T^{(p)}_{\mathfrak{p}}V_Pf) = a((0),V_Pf)[\mathfrak{p}] = a((0),f)[\mathfrak{p}P^{-1}].$$

If $\mathfrak{p} \mid P$ this simplifies and equals $a(\mathfrak{r},V_{P/\mathfrak{p}}f)$, as claimed. If $\mathfrak{p} \nmid P$ then

$$\begin{aligned}
a(\mathfrak{r},V_PT^{(1)}_{\mathfrak{p}}f) &= a(\mathfrak{r}/P,T^{(1)}_{\mathfrak{p}}f) = a(\mathfrak{r}\mathfrak{p}/P,f) + \varepsilon(\mathfrak{p})a(\mathfrak{r}/(P\mathfrak{p}),f)\\
&= a(\mathfrak{r},T^{(p)}_{\mathfrak{p}}V_Pf) + \varepsilon(\mathfrak{p})a(\mathfrak{r},V_{P\mathfrak{p}}f),
\end{aligned}$$
$$\begin{aligned}
a((0),V_PT^{(1)}_{\mathfrak{p}}f) &= a((0),T^{(1)}_{\mathfrak{p}}f)[P^{-1}] = a((0),f)[\mathfrak{p}P^{-1}] + \varepsilon(\mathfrak{p})a((0),f)[(\mathfrak{p}P)^{-1}]\\
&= a((0),T^{(p)}_{\mathfrak{p}}V_Pf) + \varepsilon(\mathfrak{p})a((0),V_{P\mathfrak{p}}f),
\end{aligned}$$

proving the claim. □

4. Galois representations

This section contains the proof of the main theorem. It is based on the Hecke operators $T^{(k)}_{\mathfrak{p}}$ for $\mathfrak{p} \mid p$ and the Frobenius operators V_P, constructed in the previous section. We start with a study of the $T^{(p)}_{\mathfrak{p}}$-action on $h \cdot f$ where f is a given non-constant $T^{(1)}_{\mathfrak{p}}$-eigenform. Next we recall the construction of the Galois representation attached to a weight 1 eigenform, by finding an ordinary eigenform of higher weight in characteristic 0 with congruent eigenvalues. Wiles' theorem describing the local behaviour at $\mathfrak{p}$ of ordinary Galois representations along with some commutative algebra allows us to prove the theorem.

Let Σ be any finite set of primes of F containing those dividing $\mathfrak{n}$, and consider the abstract Hecke algebra

$$\mathbb{T}' = \overline{\mathbb{Z}}_p[T_{\mathfrak{q}} \mid \mathfrak{q} \subset \mathfrak{o} \text{ prime }, \mathfrak{q} \notin \Sigma, \mathfrak{q} \nmid p].$$

Let ε be a fixed $\overline{\mathbb{F}}_p$-valued Hecke character of F of conductor dividing $\mathfrak{n}$.

4.1. **Hecke orbits.** Note that whereas all the forms $V_P f$, for a given non-constant $\mathbb{T}'$-eigenform f of weight 1, are common eigenvectors for $\mathbb{T}'$ sharing the same eigenvalues, the following proposition will show that $T_{\mathfrak{p}}^{(p)}$ will never act as a scalar on their $\overline{\mathbb{F}}_p$-linear span, although $\mathfrak{p}$ does not divide the level $\mathfrak{n}$. This phenomenon of failure of strong multiplicity one in characteristic p has been studied in detail by one of the authors in [38] when $F = \mathbb{Q}$ and is called "doubling". As we will see, for a general F, one has "doubling" at each prime dividing p.

Proposition 4.1. *Let $f \in M_1(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p)$ be a non-constant eigenform for $T_{\mathfrak{p}}^{(1)}$ with eigenvalue $\lambda_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$.*

(a) The elements $V_P f$, with P running through all squarefree ideals of $\mathfrak{o}$ dividing p, are linearly independent over $\overline{\mathbb{F}}_p$. Denote by W their $\overline{\mathbb{F}}_p$-linear span.

(b) For all primes $\mathfrak{p}$ dividing p, the operator $T_{\mathfrak{p}}^{(p)}$ preserves W and is annihilated by $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p})$. In particular $T_{\mathfrak{p}}^{(p)}$ is invertible on W.

(c) Fix a prime $\mathfrak{p}$ dividing p and, for all primes $\mathfrak{p}' \mid p$ distinct from $\mathfrak{p}$, fix a root $\alpha_{\mathfrak{p}'}$ of $X^2 - \lambda_{\mathfrak{p}'} X + \varepsilon(\mathfrak{p}')$. Define

$$W_{\mathfrak{p}} = W[T_{\mathfrak{p}'}^{(p)} - \alpha_{\mathfrak{p}'}\ ,\ \mathfrak{p} \neq \mathfrak{p}' \mid p]$$

as the $\overline{\mathbb{F}}_p$-subspace of W on which $T_{\mathfrak{p}'}^{(p)}$ acts by scalar multiplication by $\alpha_{\mathfrak{p}'}$ for all primes $\mathfrak{p}' \mid p$ distinct from $\mathfrak{p}$. Then $W_{\mathfrak{p}}$ has dimension 2 and $T_{\mathfrak{p}}^{(p)}$ acts on it with minimal polynomial $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p})$. In particular, if the polynomial $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p})$ has a double root, then $T_{\mathfrak{p}}^{(p)}$ does not act semi-simply on $W_{\mathfrak{p}}$.

Proof. (a) Since f is non-constant, there exists a non-zero integral ideal $\mathfrak{r}$ such that $a(\mathfrak{r}, f) \neq 0$, but $a(\mathfrak{r}', f) = 0$ for all proper divisors $\mathfrak{r}'$ of $\mathfrak{r}$. We first show that $\mathfrak{r}$ is relatively prime to (p). Indeed, if $\mathfrak{p} \mid p$ were a prime dividing $\mathfrak{r}$, then (1) would imply that

$$\lambda_{\mathfrak{p}} \cdot a(\mathfrak{r}\mathfrak{p}^{-1}, f) = a(\mathfrak{r}\mathfrak{p}^{-1}, T_{\mathfrak{p}}^{(1)} f) = a(\mathfrak{r}, f) + \varepsilon(\mathfrak{p}) a(\mathfrak{r}\mathfrak{p}^{-2}, f),$$

which is impossible since $a(\mathfrak{r}\mathfrak{p}^{-2}, f) = a(\mathfrak{r}\mathfrak{p}^{-1}, f) = 0$ and $a(\mathfrak{r}, f) \neq 0$. In fact, this argument shows that $\mathfrak{r}$ is not divisible by any prime $\mathfrak{q}$ such that there exists a Hecke operator $T_{\mathfrak{q}}^{(1)}$ acting as in (14) and such that f is a $T_{\mathfrak{q}}^{(1)}$-eigenform. One should hence expect to be able to take $\mathfrak{r} = \mathfrak{o}$, *i.e.*, to take f normalised, but we will not need this.

Suppose now that we have a linear combination $0 = \sum_{P \mid p} b_P V_P f$. We show $b_P = 0$ by induction on the number of primes dividing P. To start the induction, note that $b_{\mathfrak{o}} = 0$ as $a(\mathfrak{r}, V_P f) = a(\mathfrak{r}/P, f) = 0$ if $P \neq \mathfrak{o}$, and $a(\mathfrak{r}, V_{\mathfrak{o}} f) = a(\mathfrak{r}, f) \neq 0$. Let now $R \mid p$ be squarefree and suppose $b_Q = 0$ for all proper divisors $Q \mid R$. Then using (28) and the fact that $\mathfrak{r}$ is relatively prime to P we find

$$0 = \sum_{P \mid p} b_P a(\mathfrak{r}R, V_P f) = \sum_{P \mid p} b_P a(\mathfrak{r}R/P, f) = \sum_{P \mid R} b_P a(\mathfrak{r}R/P, f) = a(\mathfrak{r}, f) b_R,$$

hence $b_R = 0$ as claimed.

(b) By Lemma 3.7 the space W is stable under $T_{\mathfrak{p}}^{(p)}$ for all primes $\mathfrak{p}$ dividing p. For $\mathfrak{p} \mid P$ we have

$$T_{\mathfrak{p}}^{(p)2} V_P f = T_{\mathfrak{p}}^{(p)} V_{P/\mathfrak{p}} f = \lambda_{\mathfrak{p}} V_{P/\mathfrak{p}} f - \varepsilon(\mathfrak{p}) V_P f = (\lambda_{\mathfrak{p}} T_{\mathfrak{p}}^{(p)} - \varepsilon(\mathfrak{p})) V_P f,$$

whereas for $\mathfrak{p} \nmid P$

$$T_{\mathfrak{p}}^{(p)2} V_P f = T_{\mathfrak{p}}^{(p)} (\lambda_{\mathfrak{p}} V_P - \varepsilon(\mathfrak{p}) V_{P\mathfrak{p}}) f = (\lambda_{\mathfrak{p}} T_{\mathfrak{p}}^{(p)} - \varepsilon(\mathfrak{p})) V_P f,$$

showing that $T_{\mathfrak{p}}^{(p)2} - \lambda_{\mathfrak{p}} T_{\mathfrak{p}}^{(p)} + \varepsilon(\mathfrak{p}) \,\mathrm{Id}$ annihilates W.

(c) For any squarefree ideal P dividing p, we let Z_P be the $\overline{\mathbb{F}}_p$-subspace of W having basis $V_Q f$ with Q running through all divisors of P. Since f is an eigenform for $T_{\mathfrak{p}}^{(1)}$, Lemma 3.7 implies that $Z_{P\mathfrak{p}}$ is preserved by $T_{\mathfrak{p}}^{(p)}$ for all primes $\mathfrak{p} \nmid P$ and

$$Z_{P\mathfrak{p}} = Z_P \oplus T_{\mathfrak{p}}^{(p)} Z_P, \tag{29}$$

the sum being direct since $\dim(Z_{P\mathfrak{p}}) = 2\dim(Z_P)$ by (a).

Now, we show that $Z_P[T_{\mathfrak{p}'}^{(p)} - \alpha_{\mathfrak{p}'}\ ,\ \mathfrak{p}' \mid P]$ is 1-dimensional by induction on the number of primes dividing P. Note that $Z_{\mathfrak{o}}$ is a line spanned by $V_{\mathfrak{o}} f$. Suppose that $Z_P[T_{\mathfrak{p}'}^{(p)} - \alpha_{\mathfrak{p}'}\ ,\ \mathfrak{p}' \mid P] = \overline{\mathbb{F}}_p g$ and let $\mathfrak{p} \mid p$ be a prime not dividing P. By (29) one has

$$Z_{P\mathfrak{p}}[T_{\mathfrak{p}'}^{(p)} - \alpha_{\mathfrak{p}'}\ ,\ \mathfrak{p}' \mid P] = \overline{\mathbb{F}}_p g \oplus \overline{\mathbb{F}}_p T_{\mathfrak{p}}^{(p)} g,$$

on which $T_{\mathfrak{p}}^{(p)}$ does not act as a scalar, hence by (b) its minimal polynomial is $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p})$. It follows that $Z_{P\mathfrak{p}}[T_{\mathfrak{p}'}^{(p)} - \alpha_{\mathfrak{p}'}\ ,\ \mathfrak{p}' \mid P\mathfrak{p}]$ is 1-dimensional, too.

Taking P to be the product of all primes dividing p and distinct from $\mathfrak{p}$, one gets $W_{\mathfrak{p}} = \overline{\mathbb{F}}_p g \oplus \overline{\mathbb{F}}_p T_{\mathfrak{p}}^{(p)} g$, yielding the desired result. □

By counting dimensions, the above proposition implies that one can characterise W as the smallest subspace of $M_p(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p)$ containing $h \cdot f$ that is stable under $T_{\mathfrak{p}}^{(p)}$ for all $\mathfrak{p} \mid p$.

4.2. **Construction of the Galois representations.** Let $f \in M_1(\mathfrak{n}; \overline{\mathbb{F}}_p)$ be any non-zero eigenform for $\mathbb{T}'$. By Lemma 2.2, there exists an integer $k_0 > 1$ such that in weight $k = 1 + k_0(p-1)$ one has an isomorphism:

$$M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p) \otimes \overline{\mathbb{F}}_p \simeq M_k(\mathfrak{n}; \overline{\mathbb{F}}_p).$$

Since $h^{k_0} f \in M_k(\mathfrak{n}; \overline{\mathbb{F}}_p)$ is a $\mathbb{T}'$-eigenform with the same eigenvalues as f, by Deligne-Serre [10, Lemma 6.11] there exists a $\mathbb{T}'$-eigenform $g \in M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)$ the eigenvalues of which lift those of f. We define then $\rho_f : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ as the semi-simplification of the reduction modulo p of the p-adic Galois representation ρ_g with representation space $V(g)$ attached to g. If g is a cusp form, the Galois representation attached to g exists by [36] (since g is ordinary, as we show below, the existence already follows from [39]). The general case follows from the direct sum decomposition of $M_k(\mathfrak{n}, \mathbb{C})$ into the cuspidal subspace and the so-called Eisenstein subspace. The latter has a basis consisting of Eisenstein series attached to

pairs of ray class characters; these Eisenstein series are Hecke eigenforms and the direct sum of the two characters is the Galois representation attached to them (see [40, §1.5]).

By construction, ρ_f is uniquely determined by f since it is semi-simple, unramified at all $\mathfrak{q} \notin \Sigma$ not dividing p, and $\mathrm{Tr}(\rho_f(\mathrm{Frob}_\mathfrak{q}))$ equals the $T_\mathfrak{q}^{(1)}$-eigenvalue of f.

4.3. **Ordinarity.** Let $f \in M_1(\mathfrak{n}; \overline{\mathbb{F}}_p)$ be any non-constant $\mathbb{T}'$-eigenform. Since the operators $(\langle \mathfrak{q} \rangle)_{\mathfrak{q}\notin\Sigma}$ commute with the action of $\mathbb{T}'$, there exists an $\overline{\mathbb{F}}_p^\times$-valued ray class character ε of modulus $\mathfrak{n}$ and a form in $M_1(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p)$ sharing the same $\mathbb{T}'$-eigenvalues as f, that we will still denote f. Furthermore, since $\mathbb{T}'[T_\mathfrak{p}^{(1)}, \mathfrak{p} \mid p]$ acts commutatively on $M_1(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p)$, we may and do assume that f is also an eigenform for $T_\mathfrak{p}^{(1)}$ with some eigenvalue $\lambda_\mathfrak{p}$ for all primes $\mathfrak{p}$ dividing p.

For every prime $\mathfrak{p}$ dividing p, fix a root $\alpha_\mathfrak{p}$ of $X^2 - \lambda_\mathfrak{p} X + \varepsilon(\mathfrak{p})$, which is never 0. By Proposition 4.1(c), there exists a $\mathbb{T}'$-eigenform $f_\alpha \in M_p(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p)$ sharing with f the same $\mathbb{T}'$-eigenvalues such that in addition $T_\mathfrak{p}^{(p)} f_\alpha = \alpha_\mathfrak{p} f_\alpha$, for all $\mathfrak{p} \mid p$.

By the argument from the previous section, there exists a $\mathbb{T}'[T_\mathfrak{p}^{(p)}, \mathfrak{p} \mid p]$-eigenform $g \in M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)$ the eigenvalues of which lift those of f_α, in particular g is $\mathfrak{p}$-ordinary at all primes $\mathfrak{p}$ dividing p and $T_\mathfrak{p}^{(k)} g = \tilde{\alpha}_\mathfrak{p} g$ with $\tilde{\alpha}_\mathfrak{p}$ lifting $\alpha_\mathfrak{p}$. Since the operators $(\langle \mathfrak{q} \rangle)_{\mathfrak{q}\notin\Sigma}$ commute with the action of $\mathbb{T}'[T_\mathfrak{p}^{(p)}, \mathfrak{p} \mid p]$ on $\in M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)$, we can assume in addition that g has central character $\tilde{\varepsilon}$ lifting ε.

By a theorem of Wiles [39, Theorem 2] (applied to a p-stabilisation of g) for each $\mathfrak{p} \mid p$ one has a short exact sequence of $\overline{\mathbb{Q}}_p[D_\mathfrak{p}]$-modules:

$$0 \to V(g)^+ \to V(g) \to V(g)^- \to 0 \tag{30}$$

such that V_g^- has dimension 1 over $\overline{\mathbb{Q}}_p$ and the decomposition group $D_\mathfrak{p}$ of $\mathfrak{p}$ acts on it by the unramified character sending $\mathrm{Frob}_\mathfrak{p}$ to the unique unit root of $X^2 - \tilde{\alpha}_\mathfrak{p} X + \tilde{\varepsilon}(\mathfrak{p})\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{k-1}$.

Reduction modulo p yields the following result.

Proposition 4.2. *For every prime $\mathfrak{p}$ dividing p and for any root $\alpha_\mathfrak{p}$ of $X^2 - \lambda_\mathfrak{p} X + \varepsilon(\mathfrak{p})$, the representation $\rho_{f|D_\mathfrak{p}}$ admits a 1-dimensional unramified quotient on which $\mathrm{Frob}_\mathfrak{p}$ acts by $\alpha_\mathfrak{p}$.*

4.4. **Proof of the main theorem.** Let $f \in M_1(\mathfrak{n}; \overline{\mathbb{F}}_p)$ be a $\mathbb{T}'$-eigenform as in the theorem and fix a prime $\mathfrak{p}$ dividing p. As in the previous section we may and do assume that $f \in M_1(\mathfrak{n}, \varepsilon; \overline{\mathbb{F}}_p)$ for some $\overline{\mathbb{F}}_p^\times$-valued ray class character of modulus $\mathfrak{n}$ and that for all $\mathfrak{p}'$ dividing p we have $T_{\mathfrak{p}'}^{(1)} f = \lambda_{\mathfrak{p}'} f$. For all $\mathfrak{p}'$ dividing p, we choose a root $\alpha_{\mathfrak{p}'}$ of $X^2 - \lambda_{\mathfrak{p}'} X + \varepsilon(\mathfrak{p}')$.

Assume first that f is constant, *i.e.* given by the constant term of its adelic q-expansion in $\overline{\mathbb{F}}_p[\mathcal{C}\ell_F^+]$ (see §2.6). By (1), for any prime $\mathfrak{q} \notin \Sigma$, $\mathfrak{q} \nmid p$, the Hecke

operator $T_{\mathfrak{q}}^{(1)}$ sends a constant weight 1 form g to

$$T_{\mathfrak{q}}^{(1)} g = g[\mathfrak{q}] + \varepsilon(\mathfrak{q}) g[\mathfrak{q}^{-1}].$$

Now, consider the action by translation of the finite abelian group $\mathcal{C}\ell_F^+$ on its group ring $\overline{\mathbb{F}}_p[\mathcal{C}\ell_F^+]$. Since this action clearly commutes with the action of $\mathbb{T}'$, one can assume that f is an eigenform for this $\mathcal{C}\ell_F^+$-action, *i.e.* up to a non-zero scalar $f = \sum_{\mathfrak{c}\in\mathcal{C}\ell_F^+} \varphi(\mathfrak{c})[\mathfrak{c}^{-1}]$ for some character $\varphi : \mathcal{C}\ell_F^+ \to \overline{\mathbb{F}}_p^\times$. It follows then from the above equation that the $T_{\mathfrak{q}}^{(1)}$-eigenvalue of f equals $\varphi(\mathfrak{q}) + \epsilon(\mathfrak{q})\varphi^{-1}(\mathfrak{q})$. This shows that the semisimple Galois representation ρ_f is isomorphic to the Galois representation corresponding by class field theory to $\varphi \oplus \varepsilon\varphi^{-1}$. In particular, it is unramified above p.

If $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p})$ has two distinct roots $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ then Proposition 4.2 implies that $\rho_{f|D_{\mathfrak{p}}}$ admits two distinct unramified quotients on which $\mathrm{Frob}_{\mathfrak{p}}$ acts by $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$, respectively, which proves that $\rho_{f|D_{\mathfrak{p}}}$ is unramified in this case. Moreover, in this case, the trace of $\rho_f(\mathrm{Frob}_{\mathfrak{p}})$ equals $\alpha_{\mathfrak{p}} + \beta_{\mathfrak{p}} = \lambda_{\mathfrak{p}}$.

If ρ_f is a sum of two characters, then Proposition 4.2 implies that (at least) one of them is unramified at $\mathfrak{p}$. Moreover, as is well known, the determinant of ρ_f equals ε, hence is unramified in $\mathfrak{p}$, which proves that $\rho_{f|D_{\mathfrak{p}}}$ is unramified also in this case.

In the sequel we treat the remaining case where f is non-constant, ρ_f is irreducible and $X^2 - \lambda_{\mathfrak{p}} X + \varepsilon(\mathfrak{p})$ has a double root $\alpha_{\mathfrak{p}}$. Denote by $\mathfrak{m}'$ the maximal ideal of $\mathbb{T}'$ corresponding to f. Moreover, let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}'[T_{\mathfrak{p}'}^{(p)}, \mathfrak{p}' \mid p]$ corresponding to f and the choice of the $\alpha_{\mathfrak{p}'}$'s.

Let $\mathbb{T}'_k$ be the image of the abstract Hecke algebra $\mathbb{T}'$ in the ring of endomorphisms of $M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ (localisation of $M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)$ at $\mathfrak{m}$), where $k = 1 + k_0(p-1)$ is as in Lemma 2.2. In particular, the algebra generated by $\mathbb{T}'$ acting on $M_k(\mathfrak{n}; \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is naturally isomorphic to $\mathbb{T}'_k \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$.

By the q-expansion principle, $M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)$ is a torsion free $\overline{\mathbb{Z}}_p$-module, hence $\mathbb{T}'_k$ is torsion free. Moreover, $\mathbb{T}'_k$ is reduced as it is a $\overline{\mathbb{Z}}_p$-lattice in the semi-simple algebra

$$\mathbb{T}'_k \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p \simeq \prod_{g\in\mathcal{N}} \overline{\mathbb{Q}}_p,$$

where $\mathcal{N}$ denotes the set of all newforms contributing to $M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \otimes \overline{\mathbb{Q}}_p$.

To make the rest of the argument more transparent, we prove a useful fact.

Lemma 4.3. *The Hecke operator $T_{\mathfrak{p}}^{(k)}$ acting on $M_k(\mathfrak{n}; \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ does not belong to $\mathbb{T}'_k \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$. Moreover, $T_{\mathfrak{p}}^{(k)}$ acting on $M_k(\mathfrak{n}; \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ belongs to $\mathbb{T}'_k \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$, but does not belong to $\mathbb{T}'_k$.*

Proof. Since $\mathfrak{p}$ does not divide the level $\mathfrak{n}$, Strong Multiplicity One applied to each $g \in \mathcal{N}$ implies that $T_{\mathfrak{p}}^{(k)} \in \mathbb{T}'_k \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$.

By Lemma 2.2 one has a $\mathbb{T}'[T_\mathfrak{p}]$-equivariant isomorphism:

$$M_k(\mathfrak{n};\overline{\mathbb{Z}}_p)_\mathfrak{m}\otimes\overline{\mathbb{F}}_p\simeq M_k(\mathfrak{n};\overline{\mathbb{F}}_p)_\mathfrak{m}.$$

It follows that if $T_\mathfrak{p}^{(k)}$ belongs to $\mathbb{T}'_k$ acting on $M_k(\mathfrak{n};\overline{\mathbb{Z}}_p)_\mathfrak{m}$, then $T_\mathfrak{p}^{(k)}$ acting on $M_k(\mathfrak{n};\overline{\mathbb{F}}_p)_\mathfrak{m}$ belongs to $\mathbb{T}'_k\otimes\overline{\mathbb{F}}_p$, which we will show is impossible.

Define $W_\mathfrak{p}$ as in Proposition 4.1(c). Then, as one can see from q-expansions, there are $\mathbb{T}'[T_\mathfrak{p}]$-equivariant inclusions

$$M_k(\mathfrak{n};\overline{\mathbb{F}}_p)_\mathfrak{m}\supset h^{k_0-1}M_p(\mathfrak{n};\overline{\mathbb{F}}_p)_\mathfrak{m}\supset h^{k_0-1}M_p(\mathfrak{n};\overline{\mathbb{F}}_p)[\mathfrak{m}']_\mathfrak{m}\supset h^{k_0-1}W_\mathfrak{p}.$$

Hence, if $T_\mathfrak{p}^{(k)}$ belonged to $\mathbb{T}'_k\otimes\overline{\mathbb{F}}_p$, then $T_\mathfrak{p}^{(p)}$ would also belong to $\mathbb{T}'$ acting on $W_\mathfrak{p}$. However $\mathbb{T}'$ acts on $W_\mathfrak{p}$ by a character, whereas by Proposition 4.1(c) the action of $T_\mathfrak{p}^{(p)}$ on $W_\mathfrak{p}$ is not semisimple. □

Since ρ_f is irreducible, there exists by [5, Théorème 2] a free $\mathbb{T}'_k$-module $\mathcal{M}$ of rank 2 with a continuous action of G_F such that one has a G_F-equivariant isomorphism of $\overline{\mathbb{Q}}_p$-vector spaces:

$$\mathcal{M}\otimes_{\overline{\mathbb{Z}}_p}\overline{\mathbb{Q}}_p\simeq\prod_{g\in\mathcal{N}}V(g).$$

For all $\mathfrak{p}'\mid p$, the $T_{\mathfrak{p}'}^{(k)}$-eigenvalue of any $g\in\mathcal{N}$ belongs to $\overline{\mathbb{Z}}_p^\times$ (since it reduces to $\alpha_{\mathfrak{p}'}\in\overline{\mathbb{F}}_p^\times$), hence there exists a short exact sequence of $\overline{\mathbb{Q}}_p[D_\mathfrak{p}]$-modules (30) for each $g\in\mathcal{N}$.

Letting $\mathcal{M}^+=\mathcal{M}\cap\prod_{g\in\mathcal{N}}V(g)^+$ and $\mathcal{M}^-$ be the image of $\mathcal{M}$ in $\bigoplus_{g\in\mathcal{N}}V(g)^-$, one obtains a short exact sequence of $\mathbb{T}'[D_\mathfrak{p}]$-modules:

$$0\to\mathcal{M}^+\to\mathcal{M}\to\mathcal{M}^-\to 0. \tag{31}$$

Reducing (31) modulo $\mathfrak{m}'$, we get an exact sequence of $\overline{\mathbb{F}}_p[D_\mathfrak{p}]$-modules

$$\mathcal{M}^+/\mathfrak{m}'\mathcal{M}^+\to\mathcal{M}/\mathfrak{m}'\mathcal{M}\to\mathcal{M}^-/\mathfrak{m}'\mathcal{M}^-\to 0.$$

Since $\mathcal{M}^-\neq 0$, Nakayama's lemma (for the local algebra $\mathbb{T}'_k$) implies $\mathcal{M}^-/\mathfrak{m}'\mathcal{M}^-\neq 0$.

If $\dim_{\overline{\mathbb{F}}_p}(\mathcal{M}^-/\mathfrak{m}'\mathcal{M}^-)=1$, then Nakayama's lemma yields a surjective homomorphism $\mathbb{T}'_k\twoheadrightarrow\mathcal{M}^-$ of $\mathbb{T}'_k$-modules. Since $\mathbb{T}'_k$ and $\mathcal{M}^-$ have the same $\overline{\mathbb{Z}}_p$-rank (equal to half the $\overline{\mathbb{Z}}_p$-rank of $\mathcal{M}$) we deduce that $\mathcal{M}^-$ is free of rank 1 over $\mathbb{T}'_k$, in particular $\mathrm{Frob}_\mathfrak{p}$ acts on $\mathcal{M}^-$ as some element $U\in(\mathbb{T}'_k)^\times$. By (30), the image of U in $\mathbb{T}'_k\otimes\overline{\mathbb{F}}_p$ equals $T_\mathfrak{p}^{(k)}$ since for all $g\in\mathcal{N}$ the eigenvalue of U on g is the unique unit root of the Hecke polynomial $X^2-T_\mathfrak{p}^{(k)}X+\langle\mathfrak{p}\rangle\,\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{p})^{k-1}$ of g at $\mathfrak{p}$. Hence the endomorphism $T_\mathfrak{p}^{(k)}$ of $M_k(\mathfrak{n};\overline{\mathbb{F}}_p)_\mathfrak{m}$ belongs to $\mathbb{T}'_k\otimes_{\overline{\mathbb{Z}}_p}\overline{\mathbb{F}}_p$, contradicting Lemma 4.3.

Therefore $\dim_{\overline{\mathbb{F}}_p}(\mathcal{M}^-/\mathfrak{m}'\mathcal{M}^-)=2$ hence $\mathcal{M}/\mathfrak{m}'\mathcal{M}\simeq\mathcal{M}^-/\mathfrak{m}'\mathcal{M}^-$ implying that ρ_f is unramified at $\mathfrak{p}$. Moreover, since $\mathcal{M}^-/\mathfrak{m}'\mathcal{M}^-$ is a quotient of two lattices in $\prod_{g\in\mathcal{N}}V(g)^-$ and since $\mathrm{Frob}_\mathfrak{p}$ acts on each $V(g)^-$ by a scalar reducing modulo

p to $\alpha_\mathfrak{p}$, if follows that $\mathrm{Frob}_\mathfrak{p}$ acts on $\mathcal{M}/\mathfrak{m}'\mathcal{M}$ via a matrix $\left(\begin{smallmatrix}\alpha_\mathfrak{p} & * \\ 0 & \alpha_\mathfrak{p}\end{smallmatrix}\right)$, thus has trace $\lambda_\mathfrak{p} = 2\alpha_\mathfrak{p}$.

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